Risk Aversion and the Labor Margin in Dynamic Equilibrium Models

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Abstract

The household's labor margin has substantial effects on risk aversion and asset prices in dynamic equilibrium models. Even when consumption and labor are additively separable in utility, households can offset shocks to asset values or income by varying their hours of work. Empirical estimates of risk aversion from asset prices can be misleading if the labor margin is ignored. In this paper, I derive risk aversion in a standard dynamic equilibrium framework and provide closed-form and numerical solutions for risk aversion and asset prices. Asset prices are closely related to the expressions for risk aversion derived here, and are unrelated to traditional measures except in the case when labor is fixed. Results for generalized recursive preferences with labor and internal and external habits are also derived.

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1. Introduction

In a static, one-period model with utility $u(\cdot)$ defined over a single consumption good, Arrow (1964) and Pratt (1965) defined the coefficients of absolute and relative risk aversion, $-u''(c)/u'(c)$ and $-c\,u''(c)/u'(c)$. Difficulties immediately arise, however, when one attempts to generalize these concepts to the case of many periods or many goods (e.g., Kihlstrom and Mirman, 1974). These difficulties are particularly pronounced in a dynamic equilibrium model with labor, in which there is a double infinity of goods to consider—consumption and labor in every future period and state of nature—all of which may vary in response to a shock to asset returns or wealth.

The present paper shows how to compute risk aversion in dynamic equilibrium models in general. First, the paper verifies that risk aversion depends on the partial derivatives of the household's value function V with respect to wealth a—that is, the coefficients of absolute and relative risk aversion are essentially $-V_{aa}/V_a$ and $-aV_{aa}/V_a$, respectively. Even though closed-form solutions for the value function do not exist in general, the paper nevertheless derives simple, closed-form expressions for risk aversion at the model's nonstochastic steady state, or along a balanced growth path, using the fact that the derivative of the value function with respect to wealth equals the current-period marginal utility of consumption (Benveniste and Scheinkman, 1979). Importantly, these closed-form expressions seem to remain very good approximations even far away from the model's steady state.

A main result of the paper is that the household's labor margin has substantial effects on risk aversion, and hence asset prices. Even when labor and consumption are additively separable in utility, they remain connected by the household's budget constraint; in particular, the household can absorb asset return shocks either through changes in consumption, changes in hours worked, or some combination of the two. This ability to absorb shocks along either or both margins greatly alters the household's attitudes toward risk. For example, if the household's period utility is given by $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t$, the quantity $-c\,u_{11}/u_1 = \gamma$ is often referred to as the coefficient of relative risk aversion, but in fact the household is *risk neutral* with respect to gambles over asset values or wealth. Intuitively, the household is indifferent at the margin between using labor or consumption to absorb a shock to assets, and the household in this example is clearly risk neutral with respect to gambles over hours. More generally, when $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, risk aversion equals $(\gamma^{-1}+\chi^{-1})^{-1}$, a combination of the parameters on the household's consumption and labor margins, reflecting that the household absorbs shocks along both margins.[1]

While modeling risk neutrality is not a main goal of the present paper, risk neutrality nevertheless can be a desirable feature for some applications, such as labor market search or financial

[1] The intertemporal elasticity of substitution in this example is still $1/\gamma$, so a corollary of this result is that risk aversion and the intertemporal elasticity of substitution are nonreciprocal when labor supply can vary.

frictions, since it allows closed-form solutions to key features of the model.[2] Thus, an additional contribution of the present paper is to show ways of modeling risk neutrality that do not require utility to be linear in consumption, which has undesirable implications for interest rates and consumption growth. Instead, linearity of utility in any direction in the (c, l) plane is sufficient.

A final result of the paper is that risk premia computed using the Lucas-Breeden stochastic discounting framework are essentially linear in risk aversion near the nonstochastic steady state. That is, measuring risk aversion correctly—taking into account the household's labor margin—is necessary to understand asset prices in the model. Since much recent research has focused on bringing dynamic stochastic general equilibrium (DSGE) models into closer agreement with asset prices,[3] it is surprising that so little attention has been paid to measuring risk aversion correctly in these models. The present paper aims to fill that void.

There are a few previous studies that extend the Arrow-Pratt definition beyond the one-good, one-period case. In a static, multiple-good setting, Stiglitz (1969) measures risk aversion using the household's indirect utility function rather than utility itself, essentially a special case of Proposition 1 of the present paper. Constantinides (1990) measures risk aversion in a dynamic, consumption-only (endowment) economy using the household's value function, another special case of Proposition 1. Boldrin, Christiano, and Fisher (1997) apply Constantinides' definition to some very simple endowment economy models for which they can compute closed-form expressions for the value function, and hence risk aversion. The present paper builds on these studies by deriving closed-form solutions for risk aversion in dynamic equilibrium models in general, demonstrating the importance of the labor margin, and showing the tight link between risk aversion and asset prices in these models.

The remainder of the paper proceeds as follows. Section 2 defines the dynamic equilibrium framework for the analysis. Section 3 presents the main ideas of the paper, extending the definition of Arrow-Pratt risk aversion to dynamic equilibrium models with labor and deriving closed-form expressions. Section 4 demonstrates the close connection between risk aversion and asset pricing in the model. Section 5 provides numerical examples that show the accuracy and relevance of the closed-form expressions even far away from steady state. Section 6 extends the analysis to the case of balanced growth. Section 7 extends the analysis to the case of generalized recursive preferences (as in Epstein and Zin, 1989, and Weil, 1989) and habits, which have been the focus of much recent research at the boundary between macroeconomics and finance. An Appendix provides details of proofs and computations that are outlined in the main text.

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[2] See, e.g., Mortensen and Pissarides (1994) and Bernanke, Gertler, and Gilchrist (1999).

[3] See, e.g., Jermann (1998), Boldrin, Christiano, and Fisher (2001), Tallarini (2000), Uhlig (2007), Rudebusch and Swanson (2008, 2009), Van Binsbergen et al. (2010), and Backus, Routledge, and Zin (2009).

2. Dynamic Equilibrium Framework

2.1 The Household's Optimization Problem and Value Function

Time is discrete and continues forever. At each time t, the household seeks to maximize the expected present discounted value of utility flows,

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, l_\tau), \tag{1}$$

subject to the sequence of asset accumulation equations

$$a_{\tau+1} = (1+r_\tau)a_\tau + w_\tau l_\tau + d_\tau - c_\tau, \qquad \tau = t, t+1, \ldots \tag{2}$$

and the no-Ponzi-scheme condition

$$\lim_{T\to\infty} \prod_{\tau=t}^{T} (1+r_{\tau+1})^{-1} a_{T+1} \geq 0, \tag{3}$$

where E_t denotes the mathematical expectation conditional on the household's information set at time t, $\beta \in (0,1)$ is the household's discount factor, $(c_t, l_t) \in \Omega \subseteq \mathbb{R}^2$ denotes the household's consumption and labor choice in period t, a_t is the household's beginning-of-period assets, and w_t, r_t, and d_t denote the real wage, interest rate, and net transfer payments at time t. There is a finite-dimensional Markovian state vector θ_t that is exogenous to the household and governs the processes for w_t, r_t, and d_t. Conditional on θ_t, the household knows the time-t values for w_t, r_t, and d_t. The state vector and information set of the household's optimization problem at each date t is thus $(a_t; \theta_t)$. Let X denote the domain of $(a_t; \theta_t)$, and let $\Gamma : X \to \Omega$ describe the set-valued correspondence of feasible choices for (c_t, l_t) for each given $(a_t; \theta_t)$.

I make the following regularity assumptions regarding the period utility function u:

Assumption 1. *The function $u : \Omega \to \mathbb{R}$ is increasing in its first argument, decreasing in its second, twice-differentiable, and strictly concave.*

In addition to Assumption 1, a few more technical conditions are required to ensure the value function for the household's optimization problem exists and satisfies the Bellman equation (Stokey and Lucas (1990), Alvarez and Stokey (1998), and Rincón-Zapatera and Rodríguez-Palmero (2003) give different sets of such sufficient conditions). The details of these conditions are tangential to the present paper, so I simply assume:

Assumption 2. *The value function $V : X \to \mathbb{R}$ for the household's optimization problem exists and satisfies the Bellman equation*

$$V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta E_t V(a_{t+1}; \theta_{t+1}), \tag{4}$$

where a_{t+1} *is given by* (2).

The same technical conditions, plus Assumption 1, guarantee the existence of a unique optimal choice for (c_t, l_t) at each point in time, given $(a_t; \theta_t)$. Let $c_t^* \equiv c^*(a_t; \theta_t)$ and $l_t^* \equiv l^*(a_t; \theta_t)$ denote the household's optimal choices of c_t and l_t as functions of the state $(a_t; \theta_t)$. Then V can be written as

$$V(a_t; \theta_t) = u(c_t^*, l_t^*) + \beta E_t V(a_{t+1}^*; \theta_{t+1}), \tag{5}$$

where $a_{t+1}^* \equiv (1 + r_t)a_t + w_t l_t^* + d_t - c_t^*$. I also assume that these solutions are interior:

Assumption 3. *For any* $(a_t; \theta_t) \in X$*, the household's optimal choice* (c_t^*, l_t^*) *exists, is unique, and lies in the interior of* $\Gamma(a_t; \theta_t)$.

Intuitively, Assumption 3 requires the partial derivatives of u to grow sufficiently large toward the boundary that only interior solutions for c_t^* and l_t^* are optimal for all $(a_t; \theta_t) \in X$.

Assumptions 1–3 guarantee that V is continuously differentiable with respect to a and satisfies the Benveniste-Scheinkman equation, but I will require slightly more than this below:

Assumption 4. *For any* $(a_t; \theta_t)$ *in the interior of* X*, the second derivative of* V *with respect to its first argument,* $V_{11}(a_t; \theta_t)$*, exists.*

Assumption 4 also implies differentiability of the optimal policy functions, c^* and l^*, with respect to a_t. Santos (1991) provides relatively mild sufficient conditions for this assumption to be satisfied; intuitively, u must be strongly concave.

2.2 Representative Household and Steady State Assumptions

Up to this point, we have considered the case of a single household in isolation, leaving the other households of the model and the production side of the economy unspecified. Implicitly, the other households and production sector jointly determine the process for θ_t (and hence w_t, r_t, and d_t), and much of the analysis below does not need to be any more specific about these processes than this. However, to move from general expressions for risk aversion to more concrete, closed-form expressions, I adopt three standard assumptions from the DSGE literature:[4]

Assumption 5. *The household is* atomistic.

Assumption 6. *The household is* representative.

Assumption 7. *The model has a* nonstochastic steady state, $x_t = x_{t+k}$ *for* $k = 1, 2, \ldots,$ *and* $x \in \{c, l, a, w, r, d, \theta\}$.

[4] Alternative assumptions about the nature of the other households in the model or the production sector may also allow for closed-form expressions for risk aversion. However, the assumptions used here are standard and thus the most natural to pursue.

Assumption 5 implies that an individual household's choices for c_t and l_t have no effect on the aggregate quantities w_t, r_t, d_t, and θ_t. Assumption 6 implies that, when the economy is at the nonstochastic steady state, any individual household finds it optimal to choose the steady-state values of c and l given a and θ. Throughout the text, a variable without its time subscript t denotes its steady-state value.[5]

It is important to note that Assumptions 6–7 do not prohibit us from offering an individual household a hypothetical gamble of the type described below. The steady state of the model serves only as a reference point around which the *aggregate* variables w, r, d, and θ and the *other households'* choices of c, l, and a can be predicted with certainty. This reference point is important because it is there that we can compute closed-form expressions for risk aversion.

Finally, many dynamic models do not have a steady state *per se,* but rather a balanced growth path. The results below carry through essentially unchanged to the case of balanced growth. For ease of exposition, I restrict attention in Sections 3–5 to a steady state, and show in Section 6 the adjustments required under the more general:

Assumption 7′. *The model has a* balanced growth path *that can be renormalized to a nonstochastic steady state after a suitable change of variables.*

3. Risk Aversion

3.1 The Coefficient of Absolute Risk Aversion

The household's risk aversion at time t generally depends on the household's state vector at time t, $(a_t; \theta_t)$. Given this state, I consider the household's aversion to a hypothetical one-shot gamble in period t of the form:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + \sigma\varepsilon_{t+1}, \tag{6}$$

where ε_{t+1} is a random variable representing the gamble, with bounded support $[\underline{\varepsilon}, \overline{\varepsilon}]$, mean zero, unit variance, independent of θ_τ for all τ, and independent of a_τ, c_τ, and l_τ for all $\tau \leq t$. A few words about (6) are in order: First, the gamble is dated $t + 1$ to clarify that its outcome is not in the household's information set at time t. Second, c_t cannot be made the subject of the gamble without substantial modifications to the household's optimization problem, because c_t is a choice variable under control of the household at time t. However, (6) is clearly equivalent to a one-shot gamble over net transfers d_t or asset returns r_t, both of which are exogenous to the household. Indeed, thinking of the gamble as being over r_t helps to illuminate the connection between (6) and

[5] I also assume that the exogenous state θ_t contains the variances of any relevant shocks to the model. Thus, $(a; \theta)$ corresponds precisely to the nonstochastic steady state, with the variances of any shocks (other than the hypothetical gamble described in the next section) equal to zero; $c(a; \theta)$ corresponds to the household's optimal consumption choice at the nonstochastic steady state, etc.

the price of risky assets, to which we will return in Section 4. As shown there, the household's aversion to the gamble in (6) is directly linked to the premium households require to hold risky assets.

Following Arrow (1964) and Pratt (1965), we can ask what one-time fee μ the household would be willing to pay in period t to avoid the gamble in (6):

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu. \tag{7}$$

The quantity μ that makes the household just indifferent between (6) and (7), for infinitesimal σ and μ, is the household's coefficient of absolute risk aversion, which I denote by R^a.[6] Formally, this corresponds to the following definition:

Definition 1. *Let $(a_t; \theta_t)$ be an interior point of X, let $\widetilde{V}(a_t; \theta_t; \sigma)$ denote the value function for the household's optimization problem inclusive of the one-shot gamble* (6)*, and let $\mu(a_t; \theta_t; \sigma)$ denote the value of μ that satisfies $V(a_t - \frac{\mu}{1+r_t}; \theta_t) = \widetilde{V}(a_t; \theta_t; \sigma)$. The household's* coefficient of absolute risk aversion *at $(a_t; \theta_t)$, denoted $R^a(a_t; \theta_t)$, is given by $R^a(a_t; \theta_t) = \lim_{\sigma \to 0} \dfrac{\mu(a_t; \theta_t; \sigma)}{\sigma^2/2}$.*

In Definition 1, $\mu(a_t; \theta_t; \sigma)$ denotes the household's "willingness to pay" to avoid a one-shot gamble of size σ in (6). As in Arrow (1964) and Pratt (1965), R^a denotes the limit of the household's willingness to pay per unit of variance as this variance becomes small. Note that $R^a(a_t; \theta_t)$ depends on the economic state because $\mu(a_t; \theta_t; \sigma)$ depends on that state. Proposition 1 shows that $\widetilde{V}(a_t; \theta_t; \sigma)$, $\mu(a_t; \theta_t; \sigma)$, and $R^a(a_t; \theta_t)$ in Definition 1 are well-defined and that $R^a(a_t; \theta_t)$ equals the "folk wisdom" value of $-V_{11}/V_1$:[7]

Proposition 1. *Let $(a_t; \theta_t)$ be an interior point of X. Given Assumptions 1–5, $\widetilde{V}(a_t; \theta_t; \sigma)$, $\mu(a_t; \theta_t; \sigma)$, and $R^a(a_t; \theta_t)$ exist and*

$$R^a(a_t; \theta_t) \;=\; \frac{-\, E_t V_{11}(a^*_{t+1}; \theta_{t+1})}{E_t V_1(a^*_{t+1}; \theta_{t+1})}\,, \tag{8}$$

where V_1 and V_{11} denote the first and second partial derivatives of V with respect to its first argument. Given Assumptions 6–7, (8) *can be evaluated at the steady state to yield*

$$R^a(a; \theta) \;=\; \frac{-V_{11}(a; \theta)}{V_1(a; \theta)}\,. \tag{9}$$

Proof: See Appendix.

[6] I defer discussion of relative risk aversion until the next subsection because defining total household wealth is complicated by the presence of human capital—that is, the household's labor income.

[7] See, e.g., Constantinides (1990), Farmer (1990), Cochrane (2001), and Flavin and Nakagawa (2008). For the more general case of Epstein-Zin (1990) preferences, equation (8) no longer holds and there is no folk wisdom; see Section 7, below, for the more general formulas corresponding to that case.

Equations (8)–(9) are essentially Constantinides' (1990) definition of risk aversion, and have obvious similarities to Arrow (1964) and Pratt (1965). Here, of course, it is the curvature of the value function V with respect to assets that matters, rather than the curvature of the period utility function u with respect to consumption.[8]

Deriving the coefficient of absolute risk aversion in Proposition 1 is simple enough, but the problem with (8)–(9) is that closed-form expressions for V do not exist in general, even for the simplest dynamic models with labor. This difficulty may help to explain the popularity of "shortcut" approaches to measuring risk aversion, notably $-u_{11}(c_t^*, l_t^*)/\ u_1(c_t^*, l_t^*)$, which has no clear relationship to (8)–(9) except in the one-good one-period case. Boldrin, Christiano, and Fisher (1997) derive closed-form solutions for V—and hence risk aversion—for some very simple, consumption-only endowment economy models, but this approach is a nonstarter for even the simplest dynamic models with labor.

I solve this problem by observing that V_1 and V_{11} often can be computed even when closed-form solutions for V cannot be. For example, the Benveniste-Scheinkman equation,

$$V_1(a_t; \theta_t) = (1 + r_t)\, u_1(c_t^*, l_t^*), \tag{10}$$

states that the marginal value of a dollar of assets equals the marginal utility of consumption times $1 + r_t$ (the interest rate appears here because beginning-of-period assets in the model generate income in period t). In (10), u_1 is a known function. Although closed-form solutions for the functions c^* and l^* are not known in general, the points c_t^* and l_t^* often are known—for example, when they are evaluated at the nonstochastic steady state, c and l. Thus, we can compute V_1 at the nonstochastic steady state by evaluating (10) at that point.

I compute V_{11} by noting that (10) holds for general a_t; hence it can be differentiated to yield

$$V_{11}(a_t; \theta_t) \;=\; (1 + r_t) \left[u_{11}(c_t^*, l_t^*)\, \frac{\partial c_t^*}{\partial a_t} \;+\; u_{12}(c_t^*, l_t^*)\, \frac{\partial l_t^*}{\partial a_t} \right]. \tag{11}$$

All that remains is to find the derivatives $\partial c_t^* / \partial a_t$ and $\partial l_t^* / \partial a_t$.

I solve for $\partial l_t^* / \partial a_t$ by differentiating the household's intratemporal optimality condition:

$$-u_2(c_t^*, l_t^*) \;=\; w_t\, u_1(c_t^*, l_t^*), \tag{12}$$

with respect to a_t, and rearranging terms to yield

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda_t \frac{\partial c_t^*}{\partial a_t}\,, \tag{13}$$

[8] Arrow (1964) and Pratt (1965) occasionally refer to utility as being defined over "money", so one could argue that they always intended for risk aversion to be measured using indirect utility or the value function.

where

$$\lambda_t \;\equiv\; \frac{w_t u_{11}(c_t^*, l_t^*) + u_{12}(c_t^*, l_t^*)}{u_{22}(c_t^*, l_t^*) + w_t u_{12}(c_t^*, l_t^*)} \;=\; \frac{u_1(c_t^*, l_t^*) u_{12}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*) u_{11}(c_t^*, l_t^*)}{u_1(c_t^*, l_t^*) u_{22}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*) u_{12}(c_t^*, l_t^*)}\,. \tag{14}$$

If consumption and leisure in period t are normal goods, then $\lambda_t > 0$, although I do not require this restriction below. It now only remains to solve for the derivative $\partial c_t^* / \partial a_t$.

Intuitively, $\partial c_t^* / \partial a_t$ should not be too difficult to compute: it is just the household's marginal propensity to consume today out of a change in assets, which we can deduce from the household's Euler equation and budget constraint. Differentiating the Euler equation,

$$u_1(c_t^*, l_t^*) \;=\; \beta E_t (1 + r_{t+1})\, u_1(c_{t+1}^*, l_{t+1}^*), \tag{15}$$

with respect to a_t yields:[9]

$$u_{11}(c_t^*, l_t^*) \frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*, l_t^*) \frac{\partial l_t^*}{\partial a_t} = \beta E_t (1 + r_{t+1}) \left[u_{11}(c_{t+1}^*, l_{t+1}^*) \frac{\partial c_{t+1}^*}{\partial a_t} + u_{12}(c_{t+1}^*, l_{t+1}^*) \frac{\partial l_{t+1}^*}{\partial a_t} \right]. \tag{16}$$

Substituting in for $\partial l_t^* / \partial a_t$ gives

$$\big(u_{11}(c_t^*, l_t^*) - \lambda_t u_{12}(c_t^*, l_t^*)\big) \frac{\partial c_t^*}{\partial a_t} \;=\; \beta E_t (1 + r_{t+1}) \big(u_{11}(c_{t+1}^*, l_{t+1}^*) - \lambda_{t+1} u_{12}(c_{t+1}^*, l_{t+1}^*)\big) \frac{\partial c_{t+1}^*}{\partial a_t}\,. \tag{17}$$

Evaluating (17) at steady state, $\beta = (1+r)^{-1}$, $\lambda_t = \lambda_{t+1} = \lambda$, and the u_{ij} cancel, giving

$$\frac{\partial c_t^*}{\partial a_t} \;=\; E_t \frac{\partial c_{t+1}^*}{\partial a_t} \;=\; E_t \frac{\partial c_{t+k}^*}{\partial a_t}\,, \quad k = 1, 2, \ldots \tag{18}$$

$$\frac{\partial l_t^*}{\partial a_t} \;=\; E_t \frac{\partial l_{t+1}^*}{\partial a_t} \;=\; E_t \frac{\partial l_{t+k}^*}{\partial a_t}\,, \quad k = 1, 2, \ldots \tag{19}$$

In other words, whatever the change in the household's consumption today, it must be the same as the expected change in consumption tomorrow, and the expected change in consumption at each future date $t + k$.[10]

The household's budget constraint is implied by asset accumulation equation (2) and no-Ponzi condition (3). Differentiating (2) with respect to a_t, evaluating at steady state, and applying (3), (18), and (19) gives

$$\frac{1+r}{r} \frac{\partial c_t^*}{\partial a_t} \;=\; (1+r) \;+\; \frac{1+r}{r}\, w\, \frac{\partial l_t^*}{\partial a_t}\,. \tag{20}$$

That is, the expected present value of changes in household consumption must equal the change in assets (times $1 + r$) plus the expected present value of changes in labor income.

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[9] By $\partial c_{t+1}^* / \partial a_t$ I mean $\frac{\partial c_{t+1}^*}{\partial a_{t+1}} \frac{d a_{t+1}^*}{d a_t} \;=\; \frac{\partial c_{t+1}^*}{\partial a_{t+1}} \left[1 + r_{t+1} + w_t \frac{\partial l_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t} \right]$, and analogously for $\partial l_{t+1}^* / \partial a_t$, $\partial c_{t+2}^* / \partial a_t$, $\partial l_{t+2}^* / \partial a_t$, etc.

[10] Note that this equality does not follow from the steady state assumption. For example, in a model with internal habits, considered in Swanson (2009), the individual household's optimal consumption response to a change in assets increases with time, even starting from steady state.

Combining (20) with (13), we can solve for $\partial c_t^*/\partial a_t$ evaluated at the steady state:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1+w\lambda}\,. \tag{21}$$

In response to a unit increase in assets, the household raises consumption in every period by the extra asset income, r—the "golden rule"—adjusted downward by the amount $1 + w\lambda$ that takes into account the household's decrease in labor income.

We can now compute the household's coefficient of absolute risk aversion. Substituting (10), (11), (13)–(14), and (21) into (9), we have proved:

Proposition 2. *Given Assumptions 1–7, the household's coefficient of absolute risk aversion $R^a(a_t;\theta_t)$ evaluated at steady state satisfies:*

$$R^a(a;\theta) = \frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda}\,, \tag{22}$$

where u_1, u_{11}, and u_{12} denote the corresponding partial derivatives of u evaluated at the steady state (c,l), and λ is given by (14) evaluated at steady state.

When there is no labor margin in the model, Proposition 2 has the following corollary:

Corollary 3. *Given Assumptions 1–7, suppose that l_t is fixed exogenously at some $\bar{l} \in \mathbb{R}$ for all t and that the household chooses c_t optimally at each t given this constraint. Then*

$$R^a(a;\theta) = \frac{-u_{11}}{u_1}\,r. \tag{23}$$

Proof: The assumptions and steps leading up to Proposition 2, adjusted to the dynamic consumption-only case, are essentially the same as the above with $\lambda_t = 0$.

Proposition 2 and Corollary 3 are remarkable. First, the household's coefficient of absolute risk aversion in (23) is just the traditional measure, $-u_{11}/u_1$, times r, which translates assets into current-period consumption.[11] In other words, for *any* period utility function u, the traditional, static measure of risk aversion is also the correct measure in the dynamic context, regardless of whether or not u is homothetic or the rest of the model is homogeneous, whether or not we can solve for V, and no matter what the functional forms of u and V.

More generally, when households have a labor margin, Proposition 2 shows that risk aversion is less than the traditional measure by the factor $1 + w\lambda$, *even when consumption and labor are additively separable in u* (i.e., $u_{12} = 0$). Even in the additively separable case, households can partially absorb shocks to income through changes in hours worked. As a result, c_t^* depends on household labor supply, so labor and consumption are indirectly connected through the budget

[11] A gamble over a lump sum of \$$X$ is equivalent here to a gamble over an annuity of \$$X/r$. Thus, even though V_{ll}/V_1 is different from u_{11}/u_1 by a factor of r, this difference is exactly the same as a change from lump-sum to annuity units. Thus, the difference in scale is essentially one of units.

constraint.[12] When $u_{12} \neq 0$, risk aversion in Proposition 2 is further attenuated or amplified by the direct interaction between consumption and labor in utility, u_{12}. Note, however, that regardless of the signs of λ and u_{12}, risk aversion is always reduced, on net, when households can vary their labor supply:

Corollary 4. *The coefficient of absolute risk aversion* (22) *is less than or equal to* (23),

$$\frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} \leq \frac{-u_{11}}{u_1} r\,. \tag{24}$$

If $r < 1$, *then* (22) *is also less than* $-u_{11}/u_1$.

Proof: Substituting in for λ and w, (22) can be written as

$$\frac{-ru_{11}}{u_1} \frac{u_{11}u_{22} - u_{12}^2}{u_{11}u_{22} - 2\frac{u_2}{u_1} u_{11}u_{12} + \left(\frac{u_2}{u_1}\right)^2 u_{11}^2} = \frac{-ru_{11}}{u_1} \frac{1}{1 + \dfrac{\left(\frac{u_2}{u_1} u_{11} - u_{12}\right)^2}{u_{11}u_{22} - u_{12}^2}}\,. \tag{25}$$

Strict concavity of u implies that $u_{11}u_{22} - u_{12}^2 > 0$, hence the right-hand side of (25) is less than or equal to $-ru_{11}/u_1$.

Since r denotes the net interest rate, $r \ll 1$ in typical calibrations, satisfying the condition at the end of Corollary 4.

The household's labor margin can have dramatic effects on risk aversion. For example, from the left-hand side of (25) it is apparent that, no matter how large is $-u_{11}/u_1$, risk aversion can be arbitrarily small as the discriminant, $u_{11}u_{22} - u_{12}^2$, approaches zero.[13] In other words, risk aversion depends on the concavity of u in all dimensions rather than just in one dimension. Even when u_{11} is very large, the household still can be risk neutral if u_{22} is small or the cross-effect u_{12} is sufficiently large. Geometrically, if there exists any direction in (c, l)-space along which u has zero curvature, the household will optimally choose to absorb shocks to income along that line, resulting in risk-neutral behavior.

I provide some more concrete examples of risk aversion calculations in Section 3.3, below, after first defining relative risk aversion.

[12] Uhlig (2007) notes that, if households have Epstein-Zin preferences, then leisure must be taken into account in pricing assets because the value function V appears in the stochastic discount factor, and V depends on leisure. The present paper makes the point that the labor margin affects asset prices even in the case of additively separable expected utility preferences, because the labor margin changes the household's consumption process. Moreover, the methods and results of the present paper differ substantially from Uhlig (2007).

[13] The denominator of (25), $u_{11}u_{22} - 2(u_2/u_1)u_{11}u_{12} + (u_2/u_1)^2 u_{11}^2$, must also not vanish, which will be true so long as $[-u_2, u_1]'$ is not in the nullspace of the Hessian of u.

3.2 The Coefficient of Relative Risk Aversion

The difference between absolute and relative risk aversion is the size of the hypothetical gamble faced by the household. If the household faces a one-shot gamble of size A_t in period t:

$$a_{t+1} = (1+r_t)a_t + w_t l_t + d_t - c_t + A_t \sigma \varepsilon_{t+1}, \tag{26}$$

or the household can pay a one-time fee $A_t\mu$ in period t to avoid this gamble, then it follows from Proposition 1 that $\lim_{\sigma\to 0} 2\mu(\sigma)/\sigma^2$ for this gamble is given by

$$\frac{-A_t E_t V_{11}(a^*_{t+1};\theta_{t+1})}{E_t V_1(a^*_{t+1};\theta_{t+1})}\,. \tag{27}$$

The natural definition of A_t, considered by Arrow (1964) and Pratt (1965), is the household's wealth at time t. The gamble in (26) is then over a fraction of the household's wealth and (27) is referred to as the household's coefficient of relative risk aversion.

In models with labor, however, household wealth can be more difficult to define because of the presence of human capital. In these models, there are two natural definitions of human capital, so I consequently define two measures of household wealth A_t and two coefficients of relative risk aversion (27). Note that the household's financial assets a_t are not a good measure of wealth A_t, since a_t for an individual household may be zero or negative.

First, when the household's time endowment $\bar{l}$ is well defined, it is most natural to let human capital be the present discounted value of the household's time endowment, $w_t\bar{l}$. Equivalently, total household wealth A_t equals the present discounted value of leisure $w_t(\bar{l}-l^*_t)$ plus consumption c^*_t, which follows from the budget constraint (2)–(3). In this case, we have:

Definition 2. *The* coefficient of relative risk aversion, *denoted $R^r(a_t;\theta_t)$, is given by* (27) *with $A_t \equiv (1+r_t)^{-1}E_t\sum_{\tau=t}^{\infty} m_{t,\tau}\big(c^*_\tau + w_\tau(\bar{l}-l^*_\tau)\big)$, the present discounted value of household consumption and leisure, where $m_{t,\tau}$ denotes the stochastic discount factor $\beta^{\tau-t}u_1(c^*_\tau,l^*_\tau)/u_1(c^*_t,l^*_t)$.*

The factor $(1+r_t)^{-1}$ in the definition expresses wealth A_t in beginning- rather than end-of-period-t units, so that in steady state $A = \big(c + w(\bar{l}-l)\big)/r$ and $R^r(a;\theta)$ is given by

$$R^r(a;\theta) \;=\; \frac{-A\,V_{11}(a;\theta)}{V_1(a;\theta)} \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{c+w(\bar{l}-l)}{1+w\lambda}\,. \tag{28}$$

Definition 2 is the most natural in most situations. However, there are cases when the household's time endowment is not well-defined, such as when $u(c_t,l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}$ and no upper bound $\bar{l}$ on l_t is specified, or $\bar{l}$ is specified but is completely arbitrary. In these cases, it is most natural to define human capital as the present discounted value of labor income, $w_t l^*_t$, or equivalently, to define total household wealth $\tilde{A}_t$ to be the present discounted value of consumption only, rather than consumption and leisure. In this case, we have:

Definition 3. *The* consumption-only coefficient of relative risk aversion, *denoted* $R^c(a_t;\theta_t)$, *is given by* (27) *with* $\tilde{A}_t \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} c_\tau^*$.

In steady state, $\tilde{A} = c/r$, and $R^c(a;\theta)$ is given by

$$R^c(a;\theta) \;=\; \frac{-\tilde{A}\, V_{11}(a;\theta)}{V_1(a;\theta)} \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{c}{1+w\lambda}\,. \tag{29}$$

Note that Corollary 4 implies $R^c(a;\theta) \leq -cu_{11}/u_1$.

Of course, (28) and (29) are related by the ratio of the two gambles, $(c + w(\bar{l} - l))/c$. Note $R^r(a;\theta)$ may be greater or less than $-cu_{11}/u_1$, depending on the size of $w(\bar{l} - l)$ relative to c.

Other definitions of relative risk aversion, corresponding to alternative definitions of wealth, are also possible, but Definitions 2–3 are the most natural for several reasons. First, both definitions reduce to the usual present discounted value of income or consumption when there is no human capital in the model. Second, both measures reduce to the traditional $-c\,u_{11}/u_1$ when there is no labor margin in the model—that is, when $\lambda = 0$. Third, in steady state the household consumes exactly the flow of income from its wealth, rA, consistent with standard permanent income theory (where one must include the value of leisure $w(\bar{l} - l)$ as part of consumption when the value of leisure is included in wealth).

Finally, note that neither measure of relative risk aversion is reciprocal to the intertemporal elasticity of substitution:

Corollary 5. *Given Assumptions 1–7, i)* $R^c(a;\theta)$ *and the intertemporal elasticity of substitution are reciprocal if and only if* $\lambda = 0$*; ii)* $R^r(a;\theta)$ *and the intertemporal elasticity of substitution are reciprocal if and only if* $\lambda = (\bar{l} - l)/c$.

Proof: The case $w = 0$ is ruled out by Assumptions 1 and 3. The intertemporal elasticity of substitution, evaluated at steady state, is given by $\big(dc_{t+1}^* - dc_t^*\big)/\big(c\, d\log(1+r_{t+1})\big)$, which equals $-u_1/\big(c(u_{11} - \lambda u_{12})\big)$ by a calculation along the lines of (17), holding w_t fixed but allowing l_t^* and l_{t+1}^* to vary endogenously. The corollary then follows by comparison to (28) and (29).

3.3 Examples

Example 1. Consider the King-Plosser-Rebelo-type (1988) period utility function

$$u(c_t, l_t) \;=\; \frac{\big(c_t^\chi\,(1-l_t)^{1-\chi}\big)^{1-\gamma}}{1-\gamma}, \tag{30}$$

where $\gamma > 0$, $\gamma \neq 1$, $\chi \in (0,1)$, and $\bar{l} = 1$. Intuitively, consumption and leisure can be regarded as a composite good in this example, so the household's coefficient of relative risk aversion with respect to gambles over income, asset values, or wealth ought to equal γ. And, in fact,

$$R^r(a;\theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{c + w(1-l)}{1+w\lambda} \;=\; \gamma. \tag{31}$$

In contrast, the traditional measure of risk aversion, $-cu_{11}/u_1 = 1-\chi+\gamma\chi$, does not equal γ except in the limit as χ approaches unity.

The consumption-only coefficient of relative risk aversion equals $c/\big(c+w(1-l)\big)$ times (31),

$$R^c(a;\theta) \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{c}{1+w\lambda} \;=\; \gamma\chi. \tag{32}$$

The consumption-only measure is less than γ because the size of the gamble is only relative to the household's lifetime consumption, as opposed to the household's total wealth inclusive of the value of leisure.

Note that as γ approaches zero—that is, as utility approaches Cobb-Douglas—the household becomes risk neutral. In this case, utility along the line $c_t = w_t(1-l_t)\chi/(1-\chi)$ is linear, so the household is risk-neutral with respect to varying consumption and leisure along that line. This is true even though period utility is curved in both the consumption and leisure directions considered separately.

Example 2. Consider the additively separable period utility function

$$u(c_t,l_t) \;=\; \frac{c_t^{1-\gamma}}{1-\gamma} \;-\; \eta\frac{l_t^{1+\chi}}{1+\chi}\,, \tag{33}$$

where γ, χ, $\eta > 0$. The traditional measure of risk aversion for these preferences is γ, but

$$R^c(a;\theta) \;=\; \frac{-cu_{11}}{u_1}\,\frac{1}{1+w\,\frac{wu_{11}}{u_{22}}} \;=\; \frac{\gamma}{1+\frac{\gamma}{\chi}\frac{wl}{c}}\,. \tag{34}$$

$R^r(a_t;\theta_t)$ is not well defined in this example because it can be made arbitrarily large just by varying the household's time endowment $\bar{l}$, so I consider only R^c.

To simplify the following discussion, I restrict attention further in this example to the special case $c \approx wl$,[14] an assumption that I make *in this paragraph only* and nowhere else in the paper. In this special case, the closed-form expression in (34) can be simplified further to

$$R^c(a;\theta) \;\approx\; \frac{1}{\frac{1}{\gamma}+\frac{1}{\chi}}\,. \tag{35}$$

Note that (35) is less than the traditional measure of risk aversion by a factor of $1+\gamma/\chi$. Thus, if $\gamma = 2$ and $\chi = 1$—parameter values that are within the range of reasonable estimates in the literature—then the household's true aversion to risk is less than the traditional measure by a factor of about three. This point is illustrated in Figure 1, which graphs the consumption-only coefficient of relative risk aversion for this example as a function of the traditional measure, γ, for several different values of χ. If χ is very large, then the bias from using the traditional measure

[14] In steady state, $c = ra + wl + d$, so $c = wl$ holds exactly if there is neither capital nor transfers in the model. In any case, $ra + d$ is typically small, since $r \approx .01$.

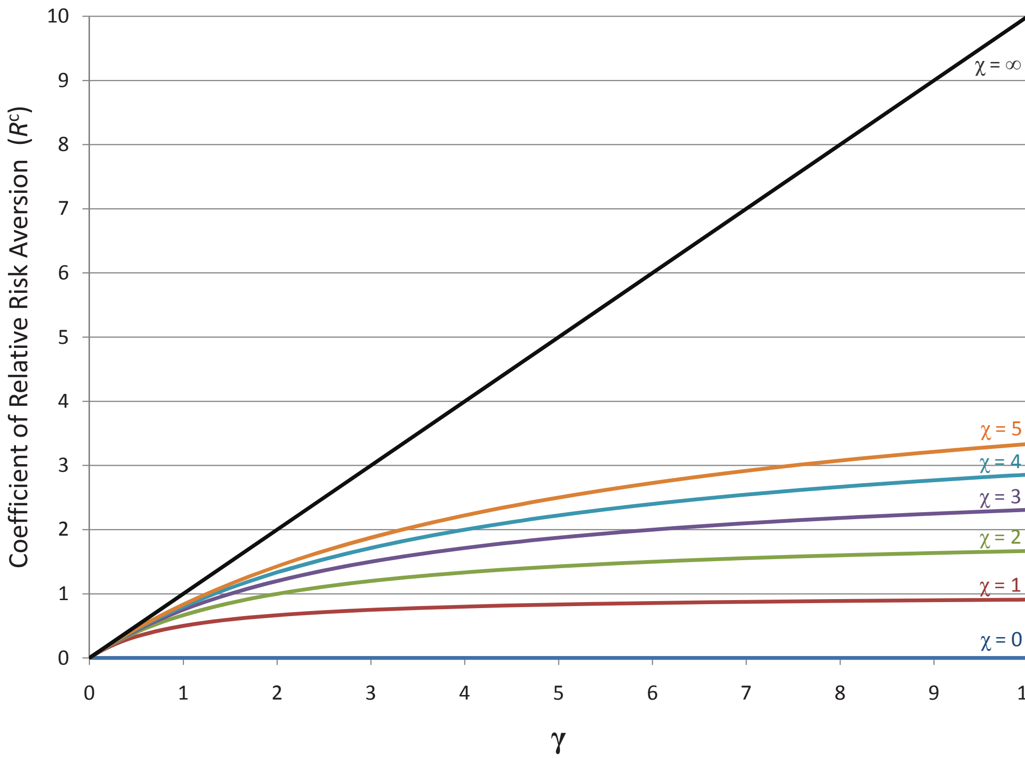

Figure 1. Coefficient of relative risk aversion $R^c(a;\theta)$ for period utility function $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ in Example 2, as a function of the traditional measure γ, for different values of χ. See text for details.

is small because household labor supply is essentially fixed.[15] However, as χ approaches 0, a common benchmark in the literature, the bias explodes and true risk aversion approaches zero—the household becomes risk neutral. Intuitively, households with linear disutility of work are risk neutral with respect to gambles over income or asset values because they can completely offset those gambles at the margin by working more or fewer hours, and households with linear disutility of work are clearly risk neutral with respect to gambles over hours.

4. Risk Aversion and Asset Pricing

In the preceding sections, I showed that the labor margin has important implications for risk aversion with respect to gambles over income or wealth. I now show that risk aversion with respect to these gambles is the right concept for asset pricing.

4.1 Measuring Risk Aversion with V As Opposed to u

Some comparison of the expressions $-V_{11}/V_1$ and $-u_{11}/u_1$ helps to clarify why the former measure is the relevant one for pricing assets, such as stocks or bonds. From Proposition 1, $-V_{11}/V_1$ is

[15] Similarly, if γ is very small, the bias from using the traditional measure is small because the household chooses to absorb income shocks almost entirely along its consumption margin. As a result, the labor margin is again almost inoperative.

the Arrow-Pratt coefficient of absolute risk aversion for gambles over income or wealth in period t. In contrast, the expression $-u_{11}/u_1$ is the risk aversion coefficient for a hypothetical gamble in which the household is *forced to consume immediately* the outcome of the gamble. Clearly, it is the former concept that corresponds to the stochastic payoffs of a standard asset such as a stock or bond. In order for $-u_{11}/u_1$ to be the relevant measure for pricing a security, it is not enough that the security pay off in units of consumption in period $t+1$. The household would additionally have to be prevented from adjusting its consumption and labor choices in period $t+1$ in response to the security's payoffs, so that the household is forced to absorb those payoffs into period $t+1$ consumption. It is difficult to imagine such a security—all standard securities in financial markets correspond to gambles over income, assets, or wealth, for which the $-V_{11}/V_1$ measure of risk aversion is the appropriate one.[16]

4.2 Risk Aversion, the Stochastic Discount Factor, and Risk Premia

Let $m_{t+1} = \beta u_1(c^*_{t+1}, l^*_{t+1})/u_1(c^*_t, l^*_t)$ denote the household's stochastic discount factor and let p_t denote the cum-dividend price of a risky asset at time t, with $E_t p_{t+1}$ normalized to unity. Define the risk premium on the asset to be the percentage difference between the risk-neutral price of the asset and its actual price,

$$\big(E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1}\big)/E_t m_{t+1} \;=\; -\mathrm{Cov}_t(dm_{t+1}, dp_{t+1})/E_t m_{t+1}, \tag{36}$$

where Cov_t denotes the covariance conditional on information at time t, and $dx_{t+1} \equiv x_{t+1} - E_t x_{t+1}$, $x \in \{m, p\}$. For small changes dc^*_{t+1} and dl^*_{t+1}, we have, to first order:

$$dm_{t+1} = \frac{\beta}{u_1(c^*_t, l^*_t)} \big[u_{11}(c^*_{t+1}, l^*_{t+1}) dc^*_{t+1} + u_{12}(c^*_{t+1}, l^*_{t+1}) dl^*_{t+1}\big]. \tag{37}$$

In (37), the household's labor margin affects m_{t+1} and hence asset prices for two reasons: First, if $u_{12} \neq 0$, changes in l_{t+1} directly affect the household's marginal utility of consumption. Second, even if $u_{12} = 0$, the presence of the labor margin affects how the household responds to shocks and hence affects dc^*_{t+1}.

Intuitively, one can already see the relationship between risk aversion and dm_{t+1} in (37): if $dl^*_{t+1} = -\lambda dc^*_{t+1}$ and $dc^*_{t+1} = r da_{t+1}/(1 + w\lambda)$, as in Section 3, then $dm_{t+1} = R^a(a;\theta) da_{t+1}$. In actuality, the relationship is more complicated than this because θ (and hence w, r, and d) may

[16] Here and throughout the paper, I take it as given that the gambles of interest are those that occur most frequently in the literature: namely, gambles over income, wealth, or asset returns (either real or nominal), for which Definitions 1–3 are the "correct" or "appropriate" measures of risk aversion. However, the reader should bear in mind that for other gambles—such as one that the household is forced to absorb entirely in current-period consumption—alternative measures of risk aversion such as the traditional $-u_{11}/u_1$ may be appropriate instead. Thus, the terms "correct" or "appropriate" in the present paper should be thought of as having the qualifier "for gambles over income, wealth, or asset returns."

change as well as a in response to macroeconomic shocks. For example, differentiating (12) and evaluating at steady state implies

$$dl^*_{t+1} = -\lambda dc^*_{t+1} - \frac{u_1}{u_{22} + wu_{12}}\, dw_{t+1} \tag{38}$$

to first order. The expression for dc^*_{t+1} is somewhat more complicated:

Lemma 6. *Given Assumptions 1–7,*

$$\begin{aligned} dc^*_{t+1} = \frac{r}{1+w\lambda}\left[da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(l\, dw_{t+k} + dd_{t+k} + a dr_{t+k}\right)\right] \\ + \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2}\, dw_{t+1} \;+\; \frac{-u_1}{u_{11} - \lambda u_{12}}\, E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(\frac{r\lambda}{1+w\lambda}\, dw_{t+k} - \beta dr_{t+k+1}\right) \end{aligned} \tag{39}$$

to first order, evaluated at the steady state.

Proof: The expression follows from (2), (3), and (15). See the Appendix for details.

For the Arrow-Pratt one-shot gamble considered in Section 3, the aggregate variables w, r, and d were held constant, so (38)–(39) reduced to (13) and (21). The term in square brackets in (39) describes the change in the present value of household income, and thus the first line of (39) describes the income effect on consumption. The last line of (39) describes the substitution effect: changes in consumption due to changes in current and future wages and interest rates. (Recall $-u_1/\big(c(u_{11} - \lambda u_{12})\big)$ is the intertemporal elasticity of substitution.)

We are now in a position to relate risk aversion to asset prices and risk premia:

Proposition 7. *Given Assumptions 1–7, the household's stochastic discount factor satisfies*

$$\begin{aligned} dm_{t+1} \;=\; -\beta R^a(a;\theta)\left[da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(l\, dw_{t+k} + dd_{t+k} + a dr_{t+k}\right)\right] \\ -\beta\, E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(\frac{r\lambda}{1+w\lambda}\, dw_{t+k} - \beta dr_{t+k+1}\right) \end{aligned} \tag{40}$$

to first order, evaluated at steady state. The risk premium in (36) *is given to second order around the steady state by*

$$R^a(a;\theta)\cdot \mathrm{Cov}_t(dp_{t+1}, d\hat{A}_{t+1}) \;+\; \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}), \tag{41}$$

where $d\hat{A}_{t+1}$ denotes the change in wealth given by the quantity in square brackets in (40) *and $d\Psi_{t+1}$ denotes the change in wages and interest rates given by the second line of* (40)*.*

Proof: Substituting (38)–(39) into (37) yields (40). Substituting (40) into (36) yields (41). Note that $\beta = E_t m_{t+1}$. Finally, $\mathrm{Cov}(dx, dy)$ is accurate to second order when dx and dy are accurate to first order.

Proposition 7 shows the importance of risk aversion for asset prices. Risk premia increase linearly with R^a near the steady state.[17] This link should not be too surprising: Propositions 1–2 describe the risk premium for the simplest gambles over household wealth, while Proposition 7 shows that the same coefficient applies to more general gambles over financial assets that may be correlated with aggregate variables such as w_t, r_t, and d_t.[18]

Proposition 7 also generalizes Merton's (1973) ICAPM to the case of variable labor. In (41), the first term is R^a times the covariance of the asset price with household wealth, while the second term captures the asset's covariance with Merton's "changes in investment opportunities". The first term can vanish if households are Arrow-Pratt risk neutral (that is, risk neutral in a cross-sectional or CAPM sense), but the second term remains nonzero because even a risk-neutral household can benefit from purchasing consumption and leisure when their prices are low—that is, when wages are low or interest rates are high. Thus, an asset that pays off well in those states of the world carries a higher price even for risk-neutral investors.

Finally, Proposition 7 implies that it is no harder or easier to match asset prices in a dynamic equilibrium model with labor than it is in such a model without labor. A given level of risk aversion in a DSGE model with labor, measured correctly, will generate just as large a risk premium as the same level of risk aversion in a DSGE model without labor, for a given set of model covariances. Thus, the equity premium is not any harder to match, or any more puzzling, in dynamic production models with endogenous labor supply than in models without it.

I conclude this section by noting that the risk premium is essentially linear in relative as well as absolute risk aversion, using an appropriate measure of covariance:

Corollary 8. *Given Assumptions 1–7, the risk premium in* (41) *can be written as*

$$R^r(a;\theta) \cdot \mathrm{Cov}_t\left(dp_{t+1}, \frac{d\hat{A}_{t+1}}{A}\right) \;+\; \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}) \tag{42}$$

or

$$R^c(a;\theta) \cdot \mathrm{Cov}_t\left(dp_{t+1}, \frac{d\hat{A}_{t+1}}{\tilde{A}}\right) \;+\; \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}), \tag{43}$$

where A and $\tilde{A}$ are as in Definitions 2–3, and $d\hat{A}_{t+1}$ and $d\Psi_{t+1}$ are as defined in Proposition 7.

Proof: See Appendix.

[17] This relationship also holds for the more general case of Epstein-Zin preferences, where it is easier to imagine varying risk aversion while holding the covariances in the model constant. See Section 7, below, and Rudebusch and Swanson (2009).

[18] Boldrin, Christiano, and Fisher (1997) argue that it is u_{11}/u_1 rather than V_{11}/V_1 that matters for the equity premium. As shown here and in the numerical example in Section 5, below, it is V_{11}/V_1—which includes the effects of the labor margin—that is crucial. Although Boldrin et al. hold R^a constant in their Section 5 and their Figure 2, the intertemporal elasticity of substitution and hence risk-free rate volatility change greatly across the circles in that figure; thus, even though R^a is held constant in their Figure 2, the covariance terms in Proposition 7 change greatly, leading to variation in the equity premium.

5. Numerical Examples

5.1 Risk Aversion Away from the Steady State

The simple, closed-form expressions for risk aversion derived above hold exactly only at the model's nonstochastic steady state. For values of $(a_t;\theta_t)$ away from steady state, these expressions are only approximations. In this section, I evaluate the accuracy of those approximations by computing risk aversion numerically for a standard real business cycle model.

There is a unit continuum of representative households in the model, each with optimization problem (1)–(3), with additively separable period utility function (33) from Example 2. The economy contains a unit continuum of perfectly competitive firms, each with production function $y_t = A_t k_t^{1-\alpha} l_t^{\alpha}$, where y_t, l_t, and k_t denote firm output, labor, and beginning-of-period capital, and A_t denotes an exogenous technology process that follows $\log A_t = \rho \log A_{t-1} + \varepsilon_t$, where ε_t is i.i.d. with mean zero and variance σ_ε^2. Labor and capital are supplied by households at the competitive wage and rental rates w_t and r_t^k. Capital is the only asset, which households accumulate according to $k_{t+1} = (1+r_t)k_t + w_t l_t - c_t$, where $r_t = r_t^k - \delta$, δ is the capital depreciation rate, and c_t denotes household consumption.

I set $\beta = .99$, $\gamma = 2$, and $\chi = 1.5$, corresponding to an intertemporal elasticity of substitution of 0.5 and Frisch elasticity of 2/3. I set $\eta = .4514$ to normalize steady-state labor $l = 1$. I set $\alpha = .7$, $\delta = .025$, $\rho = .9$, and $\sigma_\varepsilon = .01$.

The state variables of the model are k_t and A_t.[19] At the steady state, relative risk aversion is given by (34), which for the parameter values above implies $R^c(k, A) = .9145$, less than half the traditional measure of $\gamma = 2$. Away from steady state, (8) and (10)–(17) remain valid, and I use them to compute $R^c(k_t, A_t)$ by solving for V_1, V_{11}, λ_t, and $\partial c_t^*/\partial a_t$ numerically (see the Appendix for details). Figure 2 graphs the result over a wide range of values for k_t and A_t, ± 50 log percentage points (equal to about 15 and 20 standard deviations of $\log k_t$ and $\log A_t$, respectively). The solid red lines in the figure depict the solution for $R^c(k_t, A_t)$, while the horizontal dashed black lines depict the constant $R^c(k, A) = .9145$ for comparison. The key observation is that, even over the very wide range of values for (k_t, A_t) considered, the household's coefficient of relative risk aversion ranges between .88 and .94, very close to the steady-state value of .9145, and never near the traditional value of 2.[20] Thus, the closed-form expressions in Section 3 provide a good

[19] The endogenous state variable is k_t, while the exogenous state variables are A_t and K_t, the aggregate capital stock. In equilibrium, $k_t = K_t$, so I write the state vector as (k_t, A_t), although it would be written as $(k_t; A_t, K_t)$ for the analysis in Section 3.

[20] The red lines do not intersect the black lines at the vertical axis because c_t^* and l_t^* evaluated at (k, A) do not equal the nonstochastic steady state values c and l due to the presence of uncertainty (e.g., precautionary savings); one can add σ_ε to the exogenous state θ_t to capture this difference formally. Also note that absolute (rather than relative) risk aversion is countercyclical with respect to both k_t and A_t, although this is not plotted due to space constraints. In Figure 2, relative risk aversion is procyclical with respect to k_t because household wealth increases with k_t, and for this example the increase in household wealth for higher k_t more than offsets the fall in absolute risk averion.

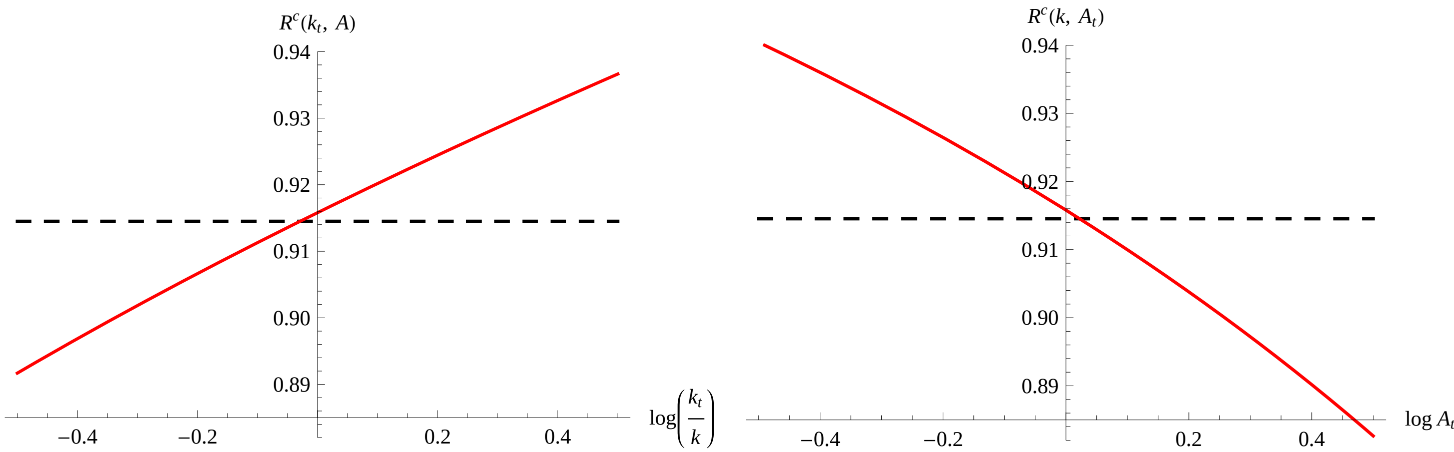

Figure 2. Coefficient of relative risk aversion $R^c(k_t, A_t)$ for period utility function $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, with $\gamma = 2$ and $\chi = 1.5$, in a standard real business cycle model. Solid red lines depict numerical solution for $R^c(k_t, A_t)$, holding one state variable at a time fixed at its steady-state value. Dashed black lines depict the constant, closed-form solution for $R^c(k, A)$ for comparison. See text for details.

approximation to the true level of risk aversion in a standard model even far away from steady state.

5.2 Risk Aversion and the Equity Premium

The numerical exercise above used parameter values that are typical of calibrations to macroeconomic data. However, it is well known that this type of parameterization produces a negligible equity premium (e.g., Mehra and Prescott, 1985, Rouwenhorst, 1994), amounting to less than one basis point for a claim to the aggregate consumption stream in the example above. In Figure 3, I consider a parameterization of the model in which the equity premium is larger, fixing $\gamma = 200$, and plot $R^c(k, A)$ and the equity premium on the left and right axes as functions of χ.[21] The equity security in this example is taken to be a claim to the aggregate consumption process, and the numerical solution procedure is the same as that used above.

As predicted in Section 4, the equity premium in Figure 3 increases essentially linearly with risk aversion. Both $R^c(k, A)$ and the equity premium fall toward zero as χ approaches zero, despite the fact that $-cu_{11}/u_1 = \gamma$ is fixed at 200.[22] These observations confirm that risk aversion as defined in the present paper—and not the traditional measure—is the proper concept for understanding asset prices in the model.

[21] For each value of χ, I set steady-state labor $l = 1$ by choosing η appropriately. See the Appendix for additional details of this computation.

[22] The simple real business cycle model in this example has difficulty matching a large equity premium even for $\gamma = 200$ and $\chi = 1000$. As noted by Rouwenhorst (1994), this is because agents with such a high value of γ endogenously choose a very smooth path for consumption. The counterfactually low consumption volatility makes matching the equity premium even more difficult than in Mehra and Prescott (1985).

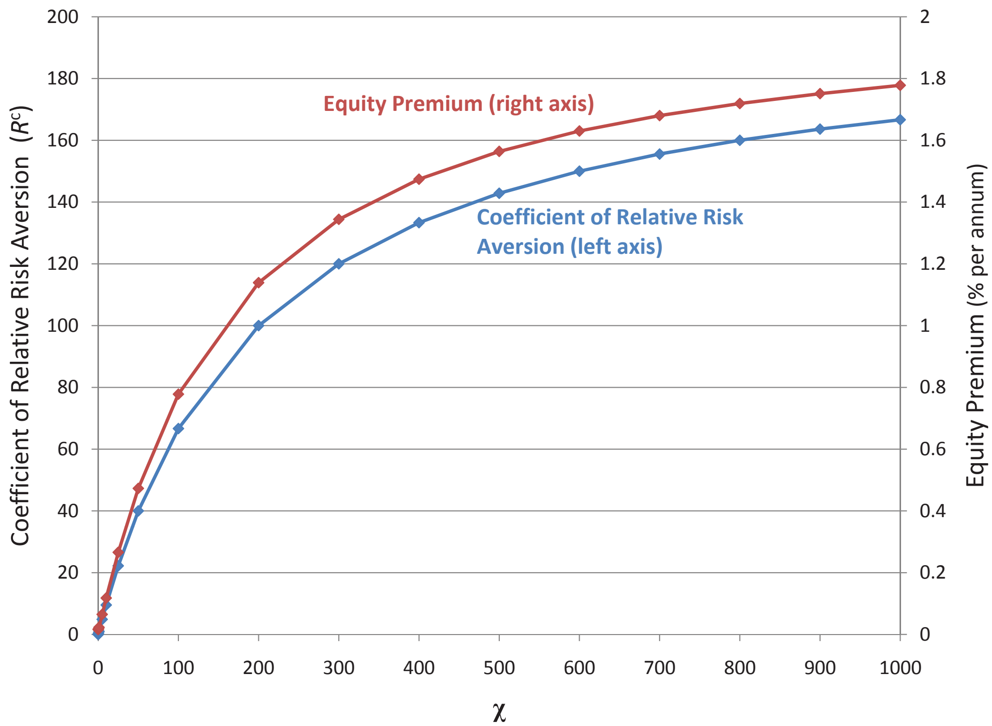

Figure 3. Coefficient of relative risk aversion $R^c(k, A)$ and the equity premium for period utility $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ with $\gamma = 200$, plotted as functions of χ, in a standard real business cycle model. The equity premium is proportional to risk aversion and both risk aversion and the equity premium approach 0 as χ approaches 0. See text for details.

6. Balanced Growth

The results in the previous sections carry through essentially unchanged to the case of balanced growth. I collect the corresponding expressions here in Lemma 9, Proposition 10, and Corollary 11, and provide proofs in the Appendix.

Along a balanced growth path, $x \in \{l, r\}$ satisfies $x_{t+k} = x_t$ for $k = 1, 2, \ldots$, and I drop the time subscript to denote the constant value. For $x \in \{a, c, w, d\}$, we have $x_{t+k} = G^k x_t$ for $k = 1, 2, \ldots$, for some $G \in (0, 1+r)$, and I let x_t^{bg} denote the balanced growth path value. I denote the balanced growth path value of θ_t by θ_t^{bg}, although the elements of θ may grow at different constant rates over time (or remain constant). Additional details regarding balanced growth are provided in King, Plosser, and Rebelo (1988, 2002).

Lemma 9. *Given Assumptions 1–6 and 7′, for $k = 1, 2, \ldots$ along the balanced growth path: i) $\lambda_{t+k}^{bg} = G^{-k}\lambda_t^{bg}$, where λ_t^{bg} denotes the balanced growth path value of λ_t, ii) $\partial c_{t+k}^*/\partial a_t = G^k \partial c_t^*/\partial a_t$, iii) $\partial l_{t+k}^*/\partial a_t = \partial l_t^*/\partial a_t$, and iv) $\partial c_t^*/\partial a_t = (1+r-G)/(1+w_t^{bg}\lambda_t^{bg})$.*

Note that $w_t^{bg}\lambda_t^{bg}$ in Lemma 9 is constant over time because w and λ grow at reciprocal rates. The larger is G, the smaller is $\partial c_t^*/\partial a_t$, since the household chooses to absorb a greater fraction of asset shocks in future periods.

Proposition 10. *Given Assumptions 1–6 and 7′, absolute risk aversion satisfies*

$$R^a(a_t^{bg};\theta_t^{bg}) \;=\; \frac{-V_{11}(a_{t+1}^{bg};\theta_{t+1}^{bg})}{V_1(a_{t+1}^{bg};\theta_{t+1}^{bg})} \tag{44}$$

and

$$R^a(a_t^{bg};\theta_t^{bg}) \;=\; \frac{-u_{11}+\lambda_t^{bg}u_{12}}{u_1}\,\frac{\frac{1+r}{G}-1}{1+w_t^{bg}\lambda_t^{bg}}\,, \tag{45}$$

where u_{ij} denotes the corresponding partial derivative of u evaluated at (c_t^{bg}, l).

Note that (45) agrees with Proposition 2 when $G = 1$. The larger is G, the smaller is R^a, since larger G implies greater household wealth and ability to absorb asset shocks.

Corollary 11. *Given Assumptions 1–6 and 7′, relative risk aversion satisfies*

$$R^r(a_t^{bg};\theta_t^{bg}) \;=\; \frac{-u_{11}+\lambda_t^{bg}u_{12}}{u_1}\,\frac{c_t^{bg}+w_t^{bg}(\bar{l}-l)}{1+w_t^{bg}\lambda_t^{bg}} \tag{46}$$

and

$$R^c(a_t^{bg};\theta_t^{bg}) \;=\; \frac{-u_{11}+\lambda_t^{bg}u_{12}}{u_1}\,\frac{c_t^{bg}}{1+w_t^{bg}\lambda_t^{bg}}\,. \tag{47}$$

Thus, the expressions for relative risk aversion are unchanged by balanced growth.

7. Generalized Recursive Preferences and Habits

Much recent research at the intersection of macroeconomics and finance considers preferences that are nonseparable across time. Here I extend the analysis of the previous sections to the case of generalized recursive preferences and internal and external consumption habits, the most common such nonseparabilities in the literature.

7.1 Generalized Recursive Preferences

The dynamic equilibrium framework in this case is the same as in Section 2, except that now I allow utility to be given by the generalized recursive specification of Epstein and Zin (1989) and Weil (1989). Thus, instead of maximizing (1), the household chooses c_t and l_t to maximize the generalized Bellman equation

$$V(a_t;\theta_t) \;=\; \max_{(c_t,l_t)\in\Gamma(a_t;\theta_t)} \; u(c_t,l_t)+\beta\left(E_t\, V(a_{t+1};\theta_{t+1})^{1-\alpha}\right)^{1/(1-\alpha)}, \tag{48}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 1$.[23] Note that (48) is the same as (4), but with the value function "twisted" and "untwisted" by the coefficient $1-\alpha$. When $\alpha = 0$, the preferences given by (48) reduce to the special case of expected utility.

For the household's optimization problem to be well-defined, I require:

Assumption 2′. *A solution $V : X \to \mathbb{R}$ to the household's generalized Bellman equation* (48) *exists.*

Technical conditions that ensure Assumption 2′ is satisfied in the case of variable labor are not yet known, but Epstein and Zin (1989) and Marinacci and Montruchhio (2010) provide some results for the consumption-only case when $u \geq 0$ everywhere.

If $u \leq 0$ everywhere, then it is natural to let $V \leq 0$ and replace (48) with

$$V(a_t;\theta_t) \;=\; \max_{(c_t,l_t)\in\Gamma(a_t;\theta_t)} \quad u(c_t,l_t) - \beta\left(E_t(-V(a_{t+1};\theta_{t+1}))^{1-\alpha}\right)^{1/(1-\alpha)}. \tag{49}$$

To avoid the possibility of complex numbers arising in the maximand of (48) or (49), I require:

Assumption 8. *Either $u : \Omega \to \mathbb{R}_+$, or $u : \Omega \to \mathbb{R}_-$.*

The main advantage of generalized recursive preferences is that they allow for greater flexibility in modeling risk aversion and the intertemporal elasticity of substitution. In (48) and (49), the intertemporal elasticity of substitution over deterministic consumption paths is exactly the same as in (4), but the household's risk aversion with respect to gambles can be amplified (or attenuated) by the additional parameter α.

7.1.1 Coefficients of Absolute and Relative Risk Aversion

Risk aversion continues to be given by Definition 1, where V is understood to mean the more general formulation in (48) or (49). The following proposition shows that risk aversion is well-defined and satisfies a generalized version of equations (8)–(9):

[23] I exclude the case $\alpha = 1$ here for simplicity. Note that, traditionally, Epstein-Zin preferences over consumption streams have been written as

$$\widetilde{V}(a_t;\theta_t) = \max_{c_t}\left[c_t^\rho + \beta\left(E_t\widetilde{V}(a_{t+1};\theta_{t+1})^{\widetilde{\alpha}}\right)^{\rho/\widetilde{\alpha}}\right]^{1/\rho},$$

but by setting $V = \widetilde{V}^\rho$ and $\alpha = 1-\widetilde{\alpha}/\rho$, this can be seen to correspond to (48). The advantage of using specification (48) rather than

$$\widetilde{V}(a_t;\theta_t) = \max_{\{c_t,l_t\}}\left[u(c_t,l_t)^\rho + \beta\left(E_t\widetilde{V}(a_{t+1};\theta_{t+1})^{\widetilde{\alpha}}\right)^{\rho/\widetilde{\alpha}}\right]^{1/\rho},$$

is that the relationship of (48) to standard dynamic programming regularity conditions, first-order conditions, and results is much more transparent. For example, (48) requires concavity of u rather than u^ρ, and the Benveniste-Scheinkman equation for (48) is the usual $V_1 = (1+r_t)u_1$ rather than $\tilde{V}_1 = (1+r_t)\tilde{V}^{(1-\rho)/\rho}u^{\rho-1}u_1$. That is, the marginal value of a dollar in (48) is just the usual marginal utility of consumption rather than something much more complicated.

Proposition 12. *Let $(a_t;\theta_t)$ be an interior point of X. Given Assumptions 1, 2′, 3–5, and 8, $\widetilde{V}(a_t;\theta_t;\sigma)$, $\mu(a_t;\theta_t;\sigma)$, and $R^a(a_t;\theta_t)$ exist and*

$$R^a(a_t;\theta_t) \;=\; \frac{-E_t V(a^*_{t+1};\theta_{t+1})^{-\alpha}\left[V_{11}(a^*_{t+1};\theta_{t+1}) \;-\; \alpha\,\dfrac{V_1(a^*_{t+1};\theta_{t+1})^2}{V(a^*_{t+1};\theta_{t+1})}\right]}{E_t V(a^*_{t+1};\theta_{t+1})^{-\alpha} V_1(a^*_{t+1};\theta_{t+1})}\,. \tag{50}$$

Given Assumptions 6–7, (50) *can be evaluated at the steady state to yield*

$$R^a(a;\theta) \;=\; \frac{-V_{11}(a;\theta)}{V_1(a;\theta)} \;+\; \alpha\,\frac{V_1(a;\theta)}{V(a;\theta)}\,. \tag{51}$$

Proof: See Appendix.

The first term in (51) is the same as the expected utility case (9), while the second term in (51) reflects the amplification or attenuation of risk aversion from the additional curvature parameter α. When $\alpha = 0$, (50)–(51) reduce to (8)–(9). When $u \geq 0$ and hence $V \geq 0$, higher values of α correspond to greater degrees of risk aversion; when u and $V \leq 0$, the opposite is true: higher values of α correspond to lesser degrees of risk aversion.

Proposition 9 is important because, unlike Proposition 1, there is no pre-existing "folk wisdom" in the profession regarding risk aversion for Epstein-Zin preferences with labor. Risk aversion for these preferences has only been computed previously in homothetic, isoelastic, consumption-only models where the value function can be computed in closed form. Proposition 9 and Proposition 10, below, do not require homotheticity of u, homogeneity of the household's optimization problem, are valid for general and unknown functional forms V, and allow for the presence of labor.

Equation (51) also highlights an important feature of risk aversion with generalized recursive preferences: it is not invariant with respect to additive shifts of the period utility function, except for the special case of expected utility ($\alpha = 0$). because the level of V enters into the right-hand side of (51). That is, the period utility functions $u(\cdot,\cdot)$ and $u(\cdot,\cdot) + k$, where k is a constant, lead to different household attitudes toward risk. The household's preferences are invariant, however, with respect to multiplicative transformations of period utility.

I now turn to computing closed-form expressions for (51). Straightforward calculcations along the lines of Section 3 show that expressions (10)–(14) and (16)–(21) for V_1, V_{11}, $\partial l^*_t/\partial a_t$, and $\partial c^*_t/\partial a_t$ remain valid, even though the Euler equation itself is slightly different.[24] Moreover, $V = u(c,l)/(1-\beta)$ at the steady state. Substituting these into (51) establishes:

Proposition 13. *Given Assumptions 1, 2′, and 3–8, the household's coefficient of absolute risk aversion $R^a(a_t;\theta_t)$ evaluated at steady state satisfies*

$$R^a(a;\theta) \;=\; \frac{-V_{11}}{V_1} + \alpha\frac{V_1}{V} \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda} \;+\; \alpha\,\frac{r\,u_1}{u}\,. \tag{52}$$

[24] The household's Euler equation is given by

$$u_1(c^*_t, l^*_t) = \beta E_t(1+r_{t+1})[V(a^*_{t+1};\theta_{t+1})/(E_t V(a^*_{t+1};\theta_{t+1})^{1-\alpha})^{1/(1-\alpha)}]^{-\alpha} u_1(c^*_{t+1}, l^*_{t+1}).$$

Relative risk aversion likewise continues to be given by Definitions 2–3, where V is understood to mean the more general formulation in (48) or (49), and where wealth is defined using the stochastic discount factor corresponding to Epstein-Zin preferences.[25] Thus, the household's coefficient of relative risk aversion, evaluated at steady state, satisfies

$$R^r(a;\theta) \;=\; \frac{-AV_{11}}{V_1} + \alpha\frac{AV_1}{V} \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{c+w(\bar{l}-l)}{1+w\lambda} \;+\; \alpha\,\frac{\big(c+w(\bar{l}-l)\big)u_1}{u}\,. \tag{53}$$

The household's consumption-only coefficient of relative risk aversion, $R^c(a;\theta)$, evaluated at steady state, is given by $c/(c+w(\bar{l}-l))$ times (53):

$$R^c(a;\theta) \;=\; \frac{-AV_{11}}{V_1} + \alpha\frac{AV_1}{V} \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{c}{1+w\lambda} \;+\; \alpha\,\frac{c\,u_1}{u}\,. \tag{54}$$

7.1.2 Examples

Example 3. Consider the additively separable period utility function

$$u(c_t,l_t) \;=\; \frac{c_t^{1-\gamma}}{1-\gamma} \;-\; \eta\frac{l_t^{1+\chi}}{1+\chi}\,, \tag{55}$$

with generalized recursive preferences (49) and $\chi>0$, $\eta>0$, and $\gamma>1$.[26] In this case, $u(\cdot,\cdot)<0$, risk aversion is decreasing in α, and $\alpha<0$ corresponds to preferences that are more risk averse than expected utility.

In models without labor, period utility $u(c_t,l_t)=c_t^{1-\gamma}/(1-\gamma)$ implies a coefficient of relative risk aversion of $\gamma+\alpha(1-\gamma)$, which I will refer to as the traditional measure.[27] The household's consumption-only coefficient of relative risk aversion (54) is given by

$$\begin{aligned} R^c(a;\theta) \;&=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{c}{1+w\lambda} \;+\; \alpha\,\frac{c\,u_1}{u} \\ &=\; \frac{\gamma}{1+\frac{\gamma}{\chi}\frac{wl}{c}} \;+\; \frac{\alpha(1-\gamma)}{1+\frac{\gamma-1}{1+\chi}\frac{wl}{c}}\,, \\ &\approx\; \frac{\gamma}{1+\frac{\gamma}{\chi}} \;+\; \frac{\alpha(1-\gamma)}{1+\frac{\gamma-1}{1+\chi}}\,, \end{aligned} \tag{56}$$

[25] The household's stochastic discount factor is given by

$$m_{t,t+1} = \beta u_1(c^*_{t+1},l^*_{t+1})[V(a^*_{t+1};\theta_{t+1})/(E_tV(a^*_{t+1};\theta_{t+1})^{1-\alpha})^{1/(1-\alpha)}]^{-\alpha}/u_1(c^*_t,l^*_t).$$

At steady state, however, this simplifies to the usual β.

[26] I restrict attention here to the case $\gamma>1$, consistent with Assumption 8. The case $\gamma\le 1$ can be considered if we place restrictions on the domain of c_t and l_t such that $u(\cdot,\cdot)<0$; one can always choose units for c_t and l_t such that this doesn't represent much of a constraint in practice. Of course, one can also consider alternative period utility functions with $\gamma\le 1$ for which $u(\cdot,\cdot)>0$.

[27] Set $\eta=0$ and $\lambda=0$ and substitute (55) into (54). This is the case, for example, in Epstein and Zin (1989) and Boldrin, Christiano, and Fisher (1997), which do not have labor. In models with variable labor, Rudebusch and Swanson (2009) refer to $\gamma+\alpha(1-\gamma)$ as the *quasi* coefficient of relative risk aversion.

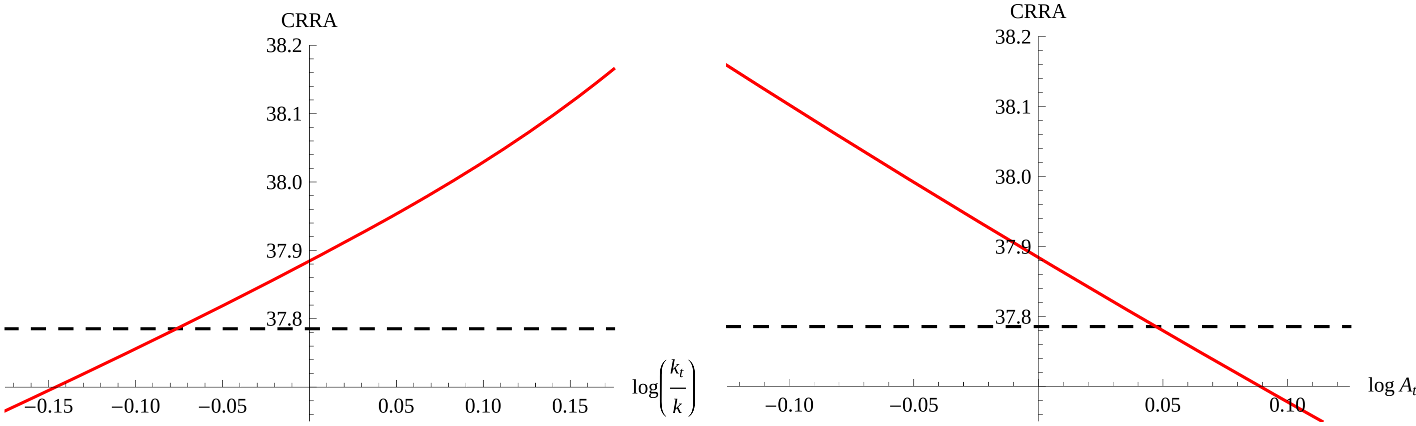

Figure 4. Coefficient of relative risk aversion $R^c(a;\theta)$ for Epstein-Zin preferences with high risk aversion and period utility $u(c_t,l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, as a function of k_t and A_t. See notes to Figure 2 and text for details.

where the last line follows if we use $c \approx wl$, as in Example 1. As in Example 1, $R^r(a;\theta)$ is not well defined in this example, so I restrict attention to $R^c(a;\theta)$.

As χ becomes large, household labor becomes less flexible and the bias from ignoring the labor margin shrinks to zero. As χ approaches zero, (56) decreases to $\alpha(1-\gamma)/\gamma$, which is close to zero if we think of γ as being close to unity. Thus, for given values of γ and α, actual risk aversion can lie anywhere between about zero and $\gamma + \alpha(1-\gamma)$, depending on χ.

In Figure 4, I extend the numerical example in Section 5 to the case of Epstein-Zin preferences with a much higher degree of risk aversion. The specification of the model and parameter values are the same as in Section 5, but with generalized recursive preferences (49) instead of expected utility (4). I set the Epstein-Zin curvature paramater $\alpha = -50$, which implies a traditional measure of risk aversion of 52, but $R^c * (a;\theta) \approx 37.8$, according to (56).

Even for the very high degree of curvature in this example, the closed-form expressions remain good approximations far away from steady state—the range of values plotted in Figure 4 corresponds to about ± 5 standard deviations of the state variables, and the coefficient of relative risk aversion ranges between about 37.6 and 38.2 over this range, very close to the steady-state value and never near the traditional measure of 52.[28]

Example 4. Van Binsbergen et al. (2010) and Backus, Routledge, and Zin (2008) consider generalized recursive preferences with

$$u(c_t,l_t) = \frac{\left(c_t^{\nu}(1-l_t)^{1-\nu}\right)^{1-\gamma}}{1-\gamma}, \tag{57}$$

[28] The unconditional standard deviations of $\log A_t$ and $\log(k_t/k)$ remain about 2.3 and 3.5 percent, respectively, and the ergodic means are 0 and about 1 percent, as before. I plot a narrower range of values for the state variables in Figure 4 than in Figure 2 because the much greater curvature of the model in this example reduces the accuracy of our numerical solution method outside this range. Also note that absolute risk aversion in this example is countercyclical in k_t and A_t, just as in the example in Section 5, but relative risk aversion is procyclical in k_t because household wealth rises with k_t by enough to offset the countercyclicality in absolute risk aversion.

where $\gamma > 0$, $\gamma \neq 1$, and $\nu \in (0,1)$. Van Binsbergen et al. call $\gamma + \alpha(1-\gamma)$ the coefficient of relative risk aversion, while Backus et al. use $\gamma\nu + \alpha(1-\gamma)\nu + (1-\nu)$, after mapping each study's notation over to the present paper's. The former measure effectively treats consumption and leisure as a composite good, while the latter is closer to the traditional measure.

Substituting (57) into (54), the household's coefficient of relative risk aversion is

$$R^r(a;\theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{c + w(1-l)}{1 + w\lambda} \;+\; \alpha\,\frac{\big(c + w(1-l)\big)u_1}{u} \;=\; \gamma + \alpha(1-\gamma). \tag{58}$$

The consumption-only coefficient of relative risk aversion satisfies[29]

$$R^c(a;\theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{c}{1 + w\lambda} \;+\; \alpha\,\frac{c\,u_1}{u} \;=\; \gamma\nu + \alpha(1-\gamma)\nu, \tag{59}$$

The former agrees with the Van Binsbergen et al. (2010) measure of risk aversion, while the latter is similar to (though not quite the same as) the Backus et al. (2008) measure. In this paper, I have provided the formal justification for both measures, (58) and (59). Note that $R^r(a;\theta)$ again corresponds to treating the Cobb-Douglas aggregate of consumption and leisure as a single, composite good.

Example 5. Tallarini (2000) considers an alternative Epstein-Zin specification

$$\tilde{V}(a_t;\theta_t) \;\equiv\; u(c_t^*, l_t^*) \;+\; \frac{\beta(1+\theta)}{(1-\beta)(1-\chi)}\,\log E_t \exp\left[\frac{(1-\beta)(1-\chi)}{1+\theta}\,\tilde{V}(a_{t+1}^*;\theta_{t+1})\right], \tag{60}$$

with period utility

$$u(c_t, l_t) \;=\; \log c_t + \theta \log(\bar{l} - l_t). \tag{61}$$

I can compute the coefficient of absolute risk aversion for (60) by following along the steps in the proof of Proposition 12, which yields

$$R^a(a;\theta) \;=\; \frac{-\tilde{V}_{11}(a;\theta)}{\tilde{V}_1(a;\theta)} \;-\; \frac{(1-\beta)(1-\chi)}{1+\theta}\,\tilde{V}_1(a;\theta). \tag{62}$$

The other steps leading up to Proposition 13 are the same, so substituting in for $\tilde{V}_1$ and $\tilde{V}_{11}$ in (62) yields a coefficient of relative risk aversion of

$$R^r(a;\theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{c + w(\bar{l}-l)}{1 + w\lambda} \;-\; \frac{1-\chi}{1+\theta}\,\big(c + w(\bar{l}-l)\big)u_1 \;=\; \chi\,. \tag{63}$$

Again, this corresponds to treating consumption and leisure as a single, composite good. The consumption-only coefficient of relative risk aversion is given by $R^c(a;\theta) = \chi/(1+\theta)$.

Both $R^r(a;\theta)$ and $R^c(a;\theta)$ differ from the value $(\chi+\theta)/(1+\theta)$ emphasized by Tallarini (2000). Tallarini applies the traditional, one-good measure of risk aversion for Epstein-Zin preferences, $\frac{-cu_{11}}{u_1} - \frac{1-\chi}{1+\theta}\,cu_1$, to the case where $\theta > 0$ but labor is fixed. This ignores the fact that, when $\theta > 0$, households will vary their labor endogenously in response to shocks.

[29] As $\nu \to 0$, $w/c \to \infty$, so consumption becomes trivial to insure with variations in labor supply. This explains why the consumption-only coefficient of relative risk aversion in (57) vanishes as $\nu \to 0$.

7.2 Habits

The second way in which nonseparabilities are typically introduced into preferences is through habits. Habits, in turn, can have substantial effects on the household's attitudes toward risk (e.g., Campbell and Cochrane, 1999, Boldrin, Christiano, and Fisher, 1997). I generalize the household's period utility function in this section to $u(c_t - h_t, l_t)$, where h_t denotes the household's reference level of consumption, or habits. I focus on additive rather than multiplicative habits because the implications for risk aversion are typically more interesting in the additive case.

7.2.1 External Habits

When the reference consumption level h_t in utility $u(c_t - h_t, l_t)$ is external to the household ("keeping up with the Joneses" utility), then the parameters that govern h_t can be incorporated into the exogenous state vector θ_t and the analysis of the previous sections carries over essentially as before. In particular, the coefficient of absolute risk aversion continues to be given by Proposition 1 in the case of expected utility and Proposition 9 for generalized recursive preferences. The household's intratemporal optimality condition (12) still implies (13)–(14), the household's Euler equation (15) still implies (16)–(19), and the household's budget constraint (2)–(3) thus implies (21), just as in Section 3.

The only real differences that arise relative to the case without habits is, first, that the steady-state point at which the derivatives of $u(\cdot,\cdot)$ are evaluated is $(c-h, l)$ rather than (c, l), and second, that relative risk aversion confronts the household with a hypothetical gamble over c rather than $c-h$, which has a tendency to make the household more risk averse for a given functional form $u(\cdot,\cdot)$, because the stakes are effectively larger.

Example 6. Consider the case of expected utility with additively separable period utility

$$u(c_t - h_t, l_t) \;=\; \frac{(c_t - h_t)^{1-\gamma}}{1-\gamma} - \eta\frac{l_t^{1+\chi}}{1+\chi}\,, \tag{64}$$

where $\gamma,\ \chi,\ \eta > 0$. The traditional measure of risk aversion for this example is $-cu_{11}/u_1 = \gamma c/(c-h)$, which exceeds γ by a factor that depends on the importance of habits relative to consumption. In contrast,

$$\begin{aligned} R^c(a;\theta) \;&=\; \frac{-cu_{11}}{u_1}\,\frac{1}{1+w\,\frac{wu_{11}}{u_{22}}}\,, \\ &=\; \frac{\gamma c}{(c-h)}\,\frac{1}{1+\frac{\gamma c}{\chi(c-h)}\frac{wl}{c}}\,. \end{aligned} \tag{65}$$

This is less than the traditional measure by the factor $1+\frac{\gamma c}{\chi(c-h)}$, using $wl \approx c$. Ignoring the labor margin in (65) thus leads to an even greater bias in the model with habits ($h > 0$) than without habits ($h = 0$). If $\gamma = 2$, $\chi = 1$, and $h = .8c$, then the household's true risk aversion is less than the traditional measure by a factor of 11.

With generalized recursive preferences rather than expected utility preferences, we have

$$R^c(a;\theta) \;=\; \frac{\gamma c}{(c-h)} \, \frac{1}{1 + \frac{\gamma c}{\chi(c-h)} \frac{wl}{c}} \;+\; \frac{\alpha(1-\gamma)c}{(c-h)} \, \frac{1}{1 + \frac{c}{(c-h)} \frac{\gamma-1}{1+\chi} \frac{wl}{c}} \,. \tag{66}$$

Again, the bias from ignoring the labor margin in (66) is even greater when $h > 0$.

7.2.2 Internal Habits

When habits are internal to the household, we must specify how the household's actions affect its future habits. I assume that the habit stock evolves according to the standard autoregressive process

$$h_t = \rho h_{t-1} + b c_{t-1}, \tag{67}$$

where $\rho \in (-1, 1)$, $b \in (0, 1)$, and $\rho + b < 1$ to ensure $h < c$ in steady state.

With internal habits, the value of h_{t+1} depends on the household's choices in period t, so I write out the dependence of the household's value function on h_t explicitly:

$$V(a_t, h_t; \theta_t) = u(c_t^* - h_t, l_t^*) + \beta \left(E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{1-\alpha}\right)^{1/(1-\alpha)}, \tag{68}$$

where $c_t^* \equiv c^*(a_t, h_t; \theta_t)$ and $l_t^* \equiv l^*(a_t, h_t; \theta_t)$ denote the household's optimal choices for consumption and labor in period t as functions of the household's state vector, and a_{t+1}^* and h_{t+1}^* denote the optimal stocks of assets and habits in period $t+1$ that are implied by c_t^* and l_t^*; that is, $a_{t+1}^* \equiv (1 + r_t)a_t + w_t l_t^* + d_t - c_t^*$ and $h_{t+1}^* \equiv \rho h_t^* + b c_t^*$. Assumptions 2–8 and Definition 1 must be modified slightly to include the additional state variable h_t, but these modifications are straightforward.

I apply Definition 1 and solve for the household's coefficient of absolute risk aversion in exactly the same manner as Propositions 1 and 12:

Proposition 14. *Let $(a_t; h_t; \theta_t)$ be an interior point of X. Given Assumptions 1, 2′, 3–5, and 8, $\widetilde{V}(a_t; h_t; \theta_t; \sigma)$, $\mu(a_t; h_t; \theta_t; \sigma)$, and $R^a(a_t; h_t; \theta_t)$ exist and*

$$R^a(a_t; h_t; \theta_t) \;=\; \frac{-E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{-\alpha} \left[V_{11}(a_{t+1}^*, h_{t+1}^*; \theta_{t+1}) \;-\; \alpha \, \dfrac{V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^2}{V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})} \right]}{E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})} \,. \tag{69}$$

Given Assumptions 6–7, (69) *can be evaluated at the steady state to yield*

$$R^a(a; h; \theta) \;=\; \frac{-V_{11}(a, h; \theta)}{V_1(a, h; \theta)} \;+\; \alpha \, \frac{V_1(a, h; \theta)}{V(a, h; \theta)} \,. \tag{70}$$

Proof: Essentially identical to the proof of Proposition 12.

Computing closed-form expressions for V_1 and V_{11} in (70) follows the same general methodology as in Section 3, but is more complicated in the presence of internal habits because of the dynamic relationship between the household's current consumption and its future habits.

Proposition 15. *Given Assumptions 1, 2′, and 3–8, the household's coefficient of absolute risk aversion $R^a(a_t; h_t; \theta_t)$ evaluated at steady state satisfies*

$$R^a(a; h; \theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{\left(1 - \frac{\beta b}{1-\beta\rho}\right) r}{1 + \left(1 - \frac{\beta b}{1-\beta\rho}\right) w\lambda} \;+\; \alpha \frac{r\, u_1}{u} \left(1 - \frac{\beta b}{1 - \beta\rho}\right). \tag{71}$$

Proof: See Appendix.

Generalizing Definitions 2 and 3, the household's coefficient of relative risk aversion, evaluated at steady state, is given by

$$R^r(a; h; \theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{\left(1 - \frac{\beta b}{1-\beta\rho}\right)\left(c + w(\bar{l} - l)\right)}{1 + \left(1 - \frac{\beta b}{1-\beta\rho}\right) w\lambda} \;+\; \alpha \frac{\left(c + w(\bar{l} - l)\right) u_1}{u} \left(1 - \frac{\beta b}{1 - \beta\rho}\right), \tag{72}$$

while the household's consumption-only coefficient of relative risk aversion is given at steady state by $c/(c + w(\bar{l} - l))$ times (72).

$$R^c(a; h; \theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{\left(1 - \frac{\beta b}{1-\beta\rho}\right) c}{1 + \left(1 - \frac{\beta b}{1-\beta\rho}\right) w\lambda} \;+\; \alpha \frac{c\, u_1}{u} \left(1 - \frac{\beta b}{1 - \beta\rho}\right). \tag{73}$$

Equations (72)–(73) have essentially the same form as the corresponding expressions in the model without habits.

Example 7. Consider the same expression for period utility as in Example 6,

$$u(c_t - h_t, l_t) \;=\; \frac{(c_t - h_t)^{1-\gamma}}{1-\gamma} \;-\; \eta \frac{l_t^{1+\chi}}{1+\chi}, \tag{74}$$

where γ, χ, $\eta > 0$, but now with habits $h_t = bc_{t-1}$ internal to the household rather than external. In this case,

$$\begin{aligned} R^c(a; h; \theta) &= \frac{-cu_{11}}{u_1} \, \frac{1 - \beta b}{1 + (1 - \beta b) w\lambda}, \\ &= \gamma \, \frac{1 - \beta b}{1 - b} \, \frac{1}{1 + \frac{\gamma}{\chi} \frac{1-\beta b}{1-b} \frac{wl}{c}}, \\ &\approx \frac{\gamma}{1 + \frac{\gamma}{\chi}}, \end{aligned} \tag{75}$$

where the last line uses $\beta \approx 1$ and $wl \approx c$.

The most striking feature of equation (75) is that it is independent of b, the importance of habits. This is in sharp contrast to the case of external habits, where risk aversion is strongly increasing in b (cf. equation (65)).

8. Conclusions

The traditional measure of risk aversion, $-cu_{11}/u_1$, ignores the household's ability to partially offset shocks to income or asset values with changes in hours worked. For reasonable parameterizations, the traditional measure can overstate risk aversion by a factor of three or more. Many studies in the macroeconomics, macro-finance, and international literatures thus may overstate the actual degree of risk aversion in their models by a substantial degree. Studies using Hansen (1985) and Rogerson (1988) linear labor preferences for algebraic simplicity are also effectively assuming risk neutrality.[30]

Risk aversion matters for asset pricing. The equity premium and risk premia on other assets are closely tied to risk aversion as defined in the present paper, and are essentially unrelated to the traditional measure, $-cu_{11}/u_1$. Risk aversion and risk premia in these models can be essentially zero even when the traditional measure of risk aversion is large.

Risk aversion and the intertemporal elasticity of substitution are nonreciprocal. This observation may be useful for model calibration since, e.g., high values of γ in $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ are not ruled out by empirical estimates of risk aversion. Empirical estimates of γ, using data on consumption and stock returns, as in Campbell (1999), do not shed any light on risk aversion unless household labor is assumed to be fixed.

It is also worth noting two non-implications of the present paper. First, the paper does not find that it is any harder or easier to match risk premia in dynamic equilibrium models with labor than in models without labor (Proposition 7). Second, the paper does not shed any light on what plausible empirical values for risk aversion might be. Empirical estimates of risk aversion based on surveys, changes in income or wealth, or cash prizes are generally just as valid in the present framework as they are in dynamic models without labor.

Finally, many of the observations of the present paper apply not just to dynamic models with labor, but to any such model with multiple goods in the utility function. Models with home production, money in the utility function, or tradeable and nontradeable goods can all imply very different household attitudes toward risk than traditional measures of risk aversion would suggest. The simple, closed-form expressions for risk aversion derived in this paper, and the methods of the paper more generally, are potentially useful in any of these cases, in pricing any asset—stocks, bonds, or futures, in foreign or domestic currency—within the framework of dynamic equilibrium models. Since these models are a mainstay of research in academia, at central banks, and international financial institutions, the applicability of the results should be widespread.

[30] Examples include Lagos and Wright (2005), Khan and Thomas (2009), Bachmann, Caballero, and Engel (2010), and Bachmann and Bayer (2009).

Appendix: Mathematical Proofs and Numerical Methods

Proof of Proposition 1

Since $(a_t;\theta_t)$ is an interior point of X, $V(a_t+\frac{\sigma\underline{\varepsilon}}{1+r_t};\theta_t)$ and $V(a_t+\frac{\sigma\overline{\varepsilon}}{1+r_t};\theta_t)$ exist for sufficiently small σ, and $V(a_t+\frac{\sigma\underline{\varepsilon}}{1+r_t};\theta_t) \leq \widetilde{V}(a_t;\theta_t;\sigma) \leq V(a_t+\frac{\sigma\overline{\varepsilon}}{1+r_t};\theta_t)$, hence $\widetilde{V}(a_t;\theta_t;\sigma)$ exists. Moreover, since $V(\cdot;\cdot)$ is continuous and increasing in its first argument, the intermediate value theorem implies there exists a unique $-\mu(a_t;\theta_t;\sigma)\in[\sigma\underline{\varepsilon},\sigma\overline{\varepsilon}]$ satisfying $V(a_t-\frac{\mu}{1+r_t};\theta_t)=\widetilde{V}(a_t;\theta_t;\sigma)$.

For a sufficiently small fee μ in (7), the change in household welfare (5) is given to first order by

$$\frac{-V_1(a_t;\theta_t)}{1+r_t}\,d\mu\,. \tag{A1}$$

Using the envelope theroem, I can rewrite (A1) as

$$-\beta E_t V_1(a^*_{t+1};\theta_{t+1})\,d\mu\,. \tag{A2}$$

Turning now to the gamble in (6), note that the household's optimal choices for consumption and labor in period t, c^*_t and l^*_t, will generally depend on the size of the gamble σ—for example, the household may undertake precautionary saving when faced with this gamble. Thus, in this section I write $c^*_t\equiv c^*(a_t;\theta_t;\sigma)$ and $l^*_t\equiv l^*(a_t;\theta_t;\sigma)$ to emphasize this dependence on σ. The household's value function, inclusive of the one-shot gamble in (6), satisfies

$$\widetilde{V}(a_t;\theta_t;\sigma)=u(c^*_t,l^*_t)+\beta E_t V(a^*_{t+1};\theta_{t+1}), \tag{A3}$$

where $a^*_{t+1}\equiv(1+r_t)a_t+w_tl^*_t+d_t-c^*_t$. Because (6) describes a one-shot gamble in period t, it affects assets a^*_{t+1} in period $t+1$ but otherwise does not affect the household's optimization problem from period $t+1$ onward; as a result, the household's value-to-go at time $t+1$ is just $V(a^*_{t+1};\theta_{t+1})$, which does not depend on σ except through a^*_{t+1}.

Differentiating (A3) with respect to σ, the first-order effect of the gamble on household welfare is

$$\left[u_1\frac{\partial c^*}{\partial\sigma}+u_2\frac{\partial l^*}{\partial\sigma}+\beta E_tV_1\cdot\left(w_t\frac{\partial l^*}{\partial\sigma}-\frac{\partial c^*}{\partial\sigma}+\varepsilon_{t+1}\right)\right]d\sigma, \tag{A4}$$

where the arguments of u_1, u_2, and V_1 are suppressed to reduce notation. Optimality of c^*_t and l^*_t implies that the terms involving $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ in (A4) cancel, as in the usual envelope theorem (these derivatives vanish at $\sigma=0$ anyway, for the reasons discussed below). Moreover, $E_tV_1(a^*_{t+1};\theta_{t+1})\varepsilon_{t+1}=0$ because ε_{t+1} is independent of θ_{t+1} and a^*_{t+1}, evaluating the latter at $\sigma=0$. Thus, the first-order cost of the gamble is zero, as in Arrow (1964) and Pratt (1965).

To second order, the effect of the gamble on household welfare is

$$\begin{aligned}\Bigg[u_{11}\left(\frac{\partial c^*}{\partial\sigma}\right)^2+2u_{12}\frac{\partial c^*}{\partial\sigma}\frac{\partial l^*}{\partial\sigma}+u_{22}\left(\frac{\partial l^*}{\partial\sigma}\right)^2+u_1\frac{\partial^2c^*}{\partial\sigma^2}+u_2\frac{\partial^2 l^*}{\partial\sigma^2}\\ +\beta E_tV_{11}\cdot\left(w_t\frac{\partial l^*}{\partial\sigma}-\frac{\partial c^*}{\partial\sigma}+\varepsilon_{t+1}\right)^2+\beta E_tV_1\cdot\left(w_t\frac{\partial^2l^*}{\partial\sigma^2}-\frac{\partial^2c^*}{\partial\sigma^2}\right)\Bigg]\frac{d\sigma^2}{2}\,.\end{aligned} \tag{A5}$$

The terms involving $\partial^2c^*/\partial\sigma^2$ and $\partial^2l^*/\partial\sigma^2$ cancel due to the optimality of c^*_t and l^*_t. The derivatives $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ vanish at $\sigma=0$ (there are two ways to see this: first, the linearized version of the model is certainty equivalent; alternatively, the gamble in (6) is isomorphic for positive and negative σ, hence c^* and l^* must be symmetric about $\sigma=0$, implying the derivatives vanish). Thus, for infinitesimal gambles, (A5) simplifies to

$$\beta E_tV_{11}(a^*_{t+1};\theta_{t+1})\,\varepsilon^2_{t+1}\frac{d\sigma^2}{2}\,. \tag{A6}$$

Finally, ε_{t+1} is independent of θ_{t+1} and a^*_{t+1}, evaluating the latter at $\sigma=0$. Since ε_{t+1} has unit variance, (A6) reduces to

$$\beta E_tV_{11}(a^*_{t+1};\theta_{t+1})\frac{d\sigma^2}{2}\,. \tag{A7}$$

Equating (A2) to (A7) allows us to solve for $d\mu$ as a function of $d\sigma^2$. Thus, $\lim_{\sigma\to 0} 2\mu(a_t;\theta_t;\sigma)/\sigma^2$ exists and is given by

$$\frac{-E_t V_{11}(a^*_{t+1};\theta_{t+1})}{E_t V_1(a^*_{t+1};\theta_{t+1})}. \tag{A8}$$

To evaluate (A8) at the nonstochastic steady state, set $a_{t+1} = a$ and $\theta_{t+1} = \theta$ to get

$$\frac{-V_{11}(a;\theta)}{V_1(a;\theta)}. \tag{A9}$$

Proof of Lemma 6

Differentiating the household's Euler equation (15) and evaluating at steady state yields

$$u_{11}(dc^*_t - E_t dc^*_{t+1}) + u_{12}(dl^*_t - E_t dl^*_{t+1}) = \beta E_t u_1 dr_{t+1}, \tag{A10}$$

which, applying (38), becomes

$$(u_{11} - \lambda u_{12})(dc^*_t - E_t dc^*_{t+1}) - \frac{u_1 u_{12}}{u_{22} + w u_{12}}(dw_t - E_t dw_{t+1}) = \beta E_t u_1 dr_{t+1}. \tag{A11}$$

Note that (A11) implies, for each $k = 1, 2, \ldots,$

$$E_t dc^*_{t+k} \;=\; dc^*_t - \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2}(dw_t - E_t dw_{t+k}) - \frac{\beta u_1}{u_{11} - \lambda u_{12}}\, E_t \sum_{i=1}^{k} dr_{t+i}. \tag{A12}$$

Combining (2)–(3), differentiating, and evaluating at steady state yields

$$E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k}(dc^*_{t+k} - w dl^*_{t+k} - l dw_{t+k} - dd_{t+k} - a dr_{t+k}) \;=\; (1+r)\, da_t. \tag{A13}$$

Substituting (38) and (A12) into (A13), and solving for dc^*_t, yields

$$\begin{aligned} dc^*_t \;=\; & \frac{r}{1+r}\,\frac{1}{1+w\lambda}\left[(1+r)da_t + E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k}\,(l\, dw_{t+k} + dd_{t+k} + a dr_{t+k})\right] \\ & + \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2}\, dw_t + \frac{1}{1+r}\,\frac{-u_1}{u_{11} - \lambda u_{12}}\, E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k}\left[\frac{r\lambda}{1+w\lambda}\, dw_{t+k} - \beta dr_{t+k+1}\right]. \end{aligned} \tag{A14}$$

Proof of Corollary 8

From Definition 2, $A_t \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau}(c^*_\tau + w_\tau(\bar{l} - l^*_\tau))$. Evaluated at steady state, $rA = c + w(\bar{l} - l)$, hence (42) follows from (41). In the same way, Definition 3 and (41) imply (43).

Numerical Solution Methods for Section 5 and Example 3

The equations of the model itself are standard:

$$Y_t = A_t K_{t-1}^{1-\alpha} L_t^{\alpha}, \tag{A15}$$

$$K_t = (1-\delta)K_{t-1} + Y_t - C_t, \tag{A16}$$

$$C_t^{-\gamma} = \beta E_t (1+r_{t+1}) C_{t+1}^{-\gamma}, \tag{A17}$$

$$\eta L_t^{\chi} / C_t^{-\gamma} = w_t, \tag{A18}$$

$$r_t = (1-\alpha) Y_t / K_{t-1} - \delta, \tag{A19}$$

$$w_t = \alpha Y_t / L_t, \tag{A20}$$

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t. \tag{A21}$$

In equations (A15)–(A21), note that K_{t-1} denotes the capital stock at the beginning of period t (or the end of period $t-1$), so the notation differs slightly from the main text for compatibility with the numerical algorithm below. To compute risk aversion, I need to append the following auxiliary variables and equations to (A15)–(A21):

$$\lambda_t = (\gamma/\chi) L_t / C_t, \tag{A22}$$

$$C_t^{-\gamma-1}\,\mathrm{DCDA}_t \;=\; \beta E_t (1+r_{t+1}) C_{t+1}^{-\gamma-1}\,\mathrm{DCDA}_{t+1}\left[(1+r_t) - (1+w_t\lambda_t)\,\mathrm{DCDA}_t\right], \tag{A23}$$

$$\mathrm{CARA}_t \;=\; E_t (1+r_{t+1}) \gamma C_{t+1}^{-\gamma-1}\,\mathrm{DCDA}_{t+1} \,/\, (C_t^{-\gamma}/\beta), \tag{A24}$$

$$\mathrm{PDVC}_t \;=\; C_t + \beta E_t (C_{t+1}^{-\gamma}/C_t^{-\gamma})\,\mathrm{PDVC}_{t+1}, \tag{A25}$$

$$\mathrm{CRRA}_t \;=\; \mathrm{CARA}_t\,\mathrm{PDVC}_t\,/(1+r_t). \tag{A26}$$

Equation (A22) corresponds to (14), (A23) to (17), (A24) to Proposition 1, and (A25)–(A26) to Definition 3. The variable DCDA_t corresponds to $\partial c_t^* / \partial a_t$. Note that

$$\frac{\partial c_{t+1}^*}{\partial a_t} \;=\; \frac{\partial c_{t+1}^*}{\partial a_{t+1}^*}\left[(1+r_t) - w_t \lambda_t \frac{\partial c_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t}\right], \tag{A27}$$

which I use in (A23). I use the envelope condition $V_1(a_t;\theta_t) \;=\; \beta(1+r_t)E_t V_1(a_{t+1};\theta_{t+1})$ to rewrite $E_t V_1(a_{t+1};\theta_{t+1})$ in (A24), and equations (10)–(11) to rewrite V_1 and V_{11} in terms of derivatives of u.

I solve (A15)–(A26) numerically using the Perturbation AIM algorithm of Swanson, Anderson, and Levin (2006) to compute second- through seventh-order Taylor series approximate solutions to (A15)–(A26) around the nonstochastic steady state. These are guaranteed to be arbitrarily accurate in a neighborhood of the nonstochastic steady state, but importantly also converge globally within the domain of convergence of the Taylor series as the order of the approximation becomes large. Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) solve a standard real business cycle model like (A15)–(A21) using a variety of numerical methods, including second- and fifth-order perturbation, and find that the perturbation solutions are among the most accurate methods globally, as well as being the fastest to compute. The perturbation solutions I compute for (A15)–(A26) are indistinguishable from one another after the third order over the range of values considered in Figure 2, consistent with Taylor series convergence, so I report only the seventh-order solution in Figure 2.

The equity premium in the model is computed as

$$p_t \;=\; \beta E_t (C_{t+1}^{-\gamma}/C_t^{-\gamma})(C_{t+1} + p_{t+1}), \tag{A28}$$

$$1/(1+r_t^f) \;=\; \beta E_t (C_{t+1}^{-\gamma}/C_t^{-\gamma}), \tag{A29}$$

$$\mathrm{ep}_t \;=\; (C_{t+1} + p_{t+1})/p_t - (1+r_t^f). \tag{A30}$$

where p_t denotes the price of equity, r_t^f the risk-free rate, and ep_t the equity premium. These equations are combined with (A15)–(A26), solved to seventh order, and evaluated at the nonstochastic steady state to produce the results in Figure 3.

For the case of Epstein-Zin preferences in Example 3, I first add equations defining the value function:

$$V_t \;=\; \frac{C_t^{1-\gamma}}{1-\gamma} \;-\; \eta\,\frac{L_t^{1+\chi}}{1+\chi} \;+\; \beta\,\mathrm{VTWIST}_t^{1/(1-\alpha)}, \tag{A31}$$

$$\mathrm{VTWIST}_t \;=\; E_t V_{t+1}^{1-\alpha}. \tag{A32}$$

Next, replace (A19) and (A27) with their Epstein-Zin counterparts:

$$C_t^{-\gamma} \;=\; \beta E_t (1+r_{t+1})\left(V_{t+1}/\mathrm{VTWIST}_t^{1/(1-\alpha)}\right)^{-\alpha} C_{t+1}^{-\gamma}, \tag{A33}$$

$$\mathrm{PDVC}_t \;=\; C_t \;+\; \beta E_t C_{t+1}^{-\gamma}/C_t^{-\gamma}\left(V_{t+1}/\mathrm{VTWIST}_t^{1/(1-\alpha)}\right)^{-\alpha}\mathrm{PDVC}_{t+1}. \tag{A34}$$

Finally, replace (A26) with the corresponding expression from Proposition 12:

$$\mathrm{CARA}_t \;=\; \frac{E_t V_{t+1}^{-\alpha}\left[(1+r_{t+1})(\gamma C_{t+1}^{-\gamma-1}\mathrm{DCDA}_{t+1}) \;+\; \alpha(1+r_{t+1})^2 C_{t+1}^{-2\gamma}/V_{t+1}\right]}{\mathrm{V1EXP}_t}\,, \tag{A35}$$

$$\mathrm{V1EXP}_t \;=\; E_t V_{t+1}^{-\alpha}(1+r_{t+1})C_{t+1}^{-\gamma}. \tag{A36}$$

The same numerical methods as above can then be applied.

Proof of Lemma 9

i) The household's Euler equation implies

$$u_1(c_t^{bg}, l) \;=\; \beta(1+r)u_1(c_{t+1}^{bg}, l) \;=\; \beta(1+r)u_1(Gc_t^{bg}, l). \tag{A37}$$

Similarly, for labor,

$$u_2(c_t^{bg}, l) \;=\; \beta(1+r)\frac{w_t^{bg}}{w_{t+1}^{bg}}u_2(c_{t+1}^{bg}, l) \;=\; \beta(1+r)G^{-1}u_2(Gc_t^{bg}, l). \tag{A38}$$

As in King, Plosser, and Rebelo (2002), I assume that preferences u are consistent with balanced growth for all initial asset stocks and wages in a neighborhood of a_t^{bg} and w_t^{bg}, and hence for all initial values of (c_t, l_t) in a neighborhood of (c_t^{bg}, l). Thus, we can differentiate (A37) and (A38) to yield

$$u_{11}(c_t^{bg}, l) \;=\; \beta(1+r)G\, u_{11}(Gc_t^{bg}, l), \tag{A39}$$

$$u_{12}(c_t^{bg}, l) \;=\; \beta(1+r)\, u_{12}(Gc_t^{bg}, l), \tag{A40}$$

$$u_{22}(c_t^{bg}, l) \;=\; \beta(1+r)G^{-1}\, u_{22}(Gc_t^{bg}, l). \tag{A41}$$

Applying (A39)–(A41) to (14),

$$\lambda_{t+1}^{bg} \;=\; \frac{w_{t+1}^{bg}u_{11}(c_{t+1}^{bg}, l) + u_{12}(c_{t+1}^{bg}, l)}{u_{22}(c_{t+1}^{bg}, l) + w_{t+1}^{bg}u_{12}(c_{t+1}^{bg}, l)} \;=\; G^{-1}\lambda_t^{bg}\,, \tag{A42}$$

ii) Assumptions 1–5 imply (10)–(17) in the text. Hence

$$\left(u_{11}(c_t^{bg}, l) - \lambda_t^{bg}u_{12}(c_t^{bg}, l)\right)\frac{\partial c_t^*}{\partial a_t} \;=\; \beta(1+r)\left(u_{11}(c_{t+1}^{bg}, l) - \lambda_{t+1}^{bg}u_{12}(c_{t+1}^{bg}, l)\right)\frac{\partial c_{t+1}^*}{\partial a_t}\,. \tag{A43}$$

Solving for $\partial c_{t+1}^*/\partial a_t$ and using (A39)–(A42) yields $\partial c_{t+1}^*/\partial a_t = G\,\partial c_t^*/\partial a_t$.
iii) Follows from (13), (A39)–(A42), and ii).
iv) Use the household's budget constraint (2)–(3) and ii) to solve for $\partial c_t^*/\partial a_t$.

Proof of Proposition 10

Proposition 1 implies (44). Assumptions 1–5 imply (10)–(17). Substituting (10), (11), (13)–(14), and Lemma 9(iv) into (44), we have

$$R^a(a_t^{bg}; \theta_t^{bg}) \;=\; \frac{-u_{11}(c_{t+1}^{bg}, l) + \lambda_{t+1}^{bg}u_{12}(c_{t+1}^{bg}, l)}{u_1(c_{t+1}^{bg}, l)}\;\frac{1+r-G}{1+w_{t+1}^{bg}\lambda_{t+1}^{bg}}\,. \tag{A44}$$

Using (A39)–(A42) and Lemma 9 completes the proof.

Proof of Corollary 11

As in Definitions 2–3, I define wealth A_t^{bg} in beginning- rather than end-of-period-t units; this requires multiplying by $(1+r)^{-1}G^{-1}$ rather than just $(1+r)^{-1}$. Then the present discounted value of consumption along the balanced growth path is given by $A_t^{bg} = c_t^{bg}/(\frac{1+r}{G} - 1)$, and the present discounted value of leisure by $w_t^{bg}(\bar{l} - l)/(\frac{1+r}{G} - 1)$.

Proof of Proposition 12

For generalized recursive preferences, the hypothetical one-shot gamble and one-time fee faced by the household are the same as for the case of expected utility. However, the household's optimality conditions for c_t^* and l_t^* (and, implicitly, a_{t+1}^*) are slightly more complicated:

$$u_1(c_t^*, l_t^*) \;=\; \beta\big(E_t V(a_{t+1}^*; \theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*; \theta_{t+1}), \tag{A45}$$
$$u_2(c_t^*, l_t^*) \;=\; -\beta w_t\big(E_t V(a_{t+1}^*; \theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*; \theta_{t+1}). \tag{A46}$$

Note that (A45) and (A46) are related by the usual $u_2(c_t^*, l_t^*) = -w_t u_1(c_t^*; l_t^*)$, and when $\alpha = 0$, (A45) and (A46) reduce to the standard optimality conditions for expected utility.

For an infinitesimal fee $d\mu$ in (7), the change in welfare for the household with generalized recursive preferences is

$$-V_1(a_t; \theta_t)\, \frac{d\mu}{1+r_t} \;=\; -\beta\big(E_t V(a_{t+1}^*; \theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*; \theta_{t+1})\, d\mu\,, \tag{A47}$$

where the right-hand side of (A47) follows from the envelope theorem.

Turning now to the gamble in (6), the first-order effect of the gamble on household welfare is

$$\left[u_1 \frac{\partial c^*}{\partial \sigma} + u_2 \frac{\partial l^*}{\partial \sigma} + \beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1 \cdot \Big(w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1}\Big)\right] d\sigma, \tag{A48}$$

where I have dropped the arguments of u_1, u_2, V, and V_1 to simplify notation. As before, optimality of c_t^* and l_t^* implies that the terms involving $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ cancel, and $E_t V^{-\alpha} V_1 \varepsilon_{t+1} = 0$ because ε_{t+1} is independent of θ_{t+1} and a_{t+1}^*, evaluating the latter at $\sigma = 0$. Thus, the first-order cost of the gamble is zero.

To second order, the effect of the gamble on household welfare is

$$\begin{aligned}
\Bigg\{ & u_{11}\left(\frac{\partial c^*}{\partial \sigma}\right)^2 + 2u_{12}\frac{\partial c^*}{\partial \sigma}\frac{\partial l^*}{\partial \sigma} + u_{22}\left(\frac{\partial l^*}{\partial \sigma}\right)^2 + u_1 \frac{\partial^2 c^*}{\partial \sigma^2} + u_2 \frac{\partial^2 l^*}{\partial \sigma^2} \\
& + \alpha\beta\big(E_t V^{1-\alpha}\big)^{(2\alpha-1)/(1-\alpha)} \left[E_t V^{-\alpha} V_1 \cdot \left(w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1}\right)\right]^2 \\
& - \alpha\beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha-1} \left[V_1 \cdot \left(w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1}\right)\right]^2 \\
& + \beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_{11} \cdot \left(w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1}\right)^2 \\
& + \beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1 \cdot \left(w_t \frac{\partial^2 l^*}{\partial \sigma^2} - \frac{\partial^2 c^*}{\partial \sigma^2}\right) \Bigg\} \frac{d\sigma^2}{2}\,.
\end{aligned} \tag{A49}$$

The derivatives $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ vanish at $\sigma = 0$, the terms involving $\partial^2 c^*/\partial\sigma^2$ and $\partial^2 l^*/\partial\sigma^2$ cancel due to the optimality of c_t^* and l_t^*, and ε_{t+1} is independent of θ_{t+1} and a_{t+1}^* (evaluating the latter at $\sigma = 0$). Thus, (A49) simplifies to

$$\beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} \left(E_t V^{-\alpha} V_{11} - \alpha E_t V^{-\alpha-1} V_1^2\right) \frac{d\sigma^2}{2}\,. \tag{A50}$$

Equating (A47) to (A50), the Arrow-Pratt coefficient of absolute risk aversion is

$$R^a(a_t; \theta_t) \;=\; \frac{-E_t V^{-\alpha} V_{11} + \alpha E_t V^{-\alpha-1} V_1^2}{E_t V^{-\alpha} V_1}\,. \tag{A51}$$

Since (A51) is already evaluated at $\sigma = 0$, to evaluate it at the nonstochastic steady state, set $a_{t+1} = a$, $\theta_{t+1} = \theta$ to get

$$R^a(a;\theta) \;=\; \frac{-V_{11}(a;\theta)}{V_1(a;\theta)} + \alpha\frac{V_1(a;\theta)}{V(a;\theta)}\,. \tag{A52}$$

Proof of Proposition 15

Readers working through this proof may find it easier to first consider the case $\alpha = \rho = 0$, that is, expected utility with one-period habits.

The household's first-order conditions for (68) with respect to consumption and labor are given by

$$u_1 \;=\; \beta\big(E_tV^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV^{-\alpha}[V_1 - bV_2], \tag{A53}$$
$$u_2 \;=\; -\beta w_t\big(E_tV^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV^{-\alpha}V_1, \tag{A54}$$

where I drop the arguments of u and V to reduce notation.

Differentiating (68) with respect to its first two arguments and applying the envelope theorem yields

$$V_1 \;=\; \beta(1+r_t)\left(E_tV^{1-\alpha}\right)^{\alpha/(1-\alpha)}E_tV^{-\alpha}V_1, \tag{A55}$$
$$V_2 \;=\; -u_1 \;+\; \rho\beta\big(E_tV^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV^{-\alpha}V_2. \tag{A56}$$

Equations (A54) and (A55) can be used to solve for V_1 in terms of current-period utility:

$$V_1(a_t,h_t;\theta_t) \;=\; -\frac{(1+r_t)}{w_t}\,u_2(c_t^* - h_t, l_t^*). \tag{A57}$$

To solve for V_{11}, differentiate (A57) with respect to a_t to yield

$$V_{11}(a_t,h_t;\theta_t) \;=\; -\frac{(1+r_t)}{w_t}\left(u_{12}\,\frac{\partial c_t^*}{\partial a_t} + u_{22}\,\frac{\partial l_t^*}{\partial a_t}\right), \tag{A58}$$

It remains to solve for $\partial c_t^*/\partial a_t$ and $\partial l_t^*/\partial a_t$, which I do in the same manner as in Section 3, except that the dynamics of internal habits require us to solve for $\partial c_\tau^*/\partial a_t$ and $\partial l_\tau^*/\partial a_t$ for all dates $\tau \geq t$ at the same time. To better keep track of these dynamics, I henceforth let a time subscript $\tau \geq t$ denote a generic future date and reserve the subscript t to denote the date of the current period—the period in which the household faces the hypothetical one-shot gamble.

I solve for $\partial l_\tau^*/\partial a_t$ in terms of $\partial c_\tau^*/\partial a_t$ in much the same manner as before, except that the expressions are more complicated due to the persistence of habits and the household's more complicated discounting of future periods. Note first that (A56) can be used to solve for V_2 in terms of current and future marginal utility:

$$V_2(a_t,h_t;\theta_t) = -(1-\rho\beta F)^{-1}\,u_1(c_t^* - h_t^*, l_t^*), \tag{A59}$$

where F denotes the "generalized recursive" forward operator; that is,

$$Fx_\tau \equiv \big(E_\tau V(a_{\tau+1}^*, h_{\tau+1}^*;\theta_{\tau+1}^*)^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_\tau V(a_{\tau+1}^*, h_{\tau+1}^*;\theta_{\tau+1}^*)^{-\alpha}x_{\tau+1}. \tag{A60}$$

The household's intratemporal optimality condition ((A54) combined with (A55)) implies

$$-u_2(c_\tau^* - h_\tau^*, l_\tau^*) \;=\; w_\tau[u_1(c_\tau^* - h_\tau^*, l_\tau^*) \;+\; b\beta E_\tau V_2(a_{\tau+1}^*, h_{\tau+1}^*;\theta_{\tau+1})]. \tag{A61}$$
$$\;=\; w_\tau\big(1-\beta bF(1-\beta\rho F)^{-1}\big)\,u_1(c_\tau^* - h_\tau^*, l_\tau^*), \tag{A62}$$

Differentiating (A62) with respect to a_t and evaluating at steady state yields

$$-u_{12}\left(\frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t}\right) - u_{22}\frac{\partial l_\tau^*}{\partial a_t} \;=\; w\left(1-\beta bF(1-\beta\rho F)^{-1}\right)\left[u_{11}\left(\frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t}\right) + u_{12}\frac{\partial l_\tau^*}{\partial a_t}\right], \tag{A63}$$

where I have used the fact that

$$\frac{\partial}{\partial a_t}\,Fx_\tau \;=\; F\,\frac{\partial x_\tau}{\partial a_t} \tag{A64}$$

when the derivative is evaluated at steady state. Solving (A63) for $\partial l_\tau^*/\partial a_t$ yields

$$\begin{aligned}\frac{\partial l_\tau^*}{\partial a_t} \;=\; &-\frac{u_{12}+wu_{11}-\beta(\rho u_{12}+(\rho+b)wu_{11})F}{u_{22}+wu_{12}}\times\\ &\left[1-\frac{\beta(\rho u_{22}+(\rho+b)wu_{12})}{u_{22}+wu_{12}}F\right]^{-1}\left(1-bL(1-\rho L)^{-1}\right)\frac{\partial c_\tau^*}{\partial a_t}\,. \qquad \text{(A65)}\end{aligned}$$

where I've used $h_\tau = bL(1-\rho L)^{-1}c_\tau$ and I assume $\left|\beta(\rho u_{22}+(\rho+b)wu_{12})/(u_{22}+wu_{12})\right| < 1$ to ensure convergence. This solves for $\partial l_t^*/\partial a_t$ in terms of (current and future) $\partial c_\tau^*/\partial a_t$.

I now turn to solving for $\partial c_\tau^*/\partial a_t$. The household's intertemporal optimality (Euler) condition is given by

$$\frac{1}{w_\tau}\,u_2(c_\tau^*-h_\tau^*,l_\tau^*) \;=\; \beta F\,\frac{1+r_\tau}{w_\tau}\,u_2(c_\tau^*-h_\tau^*,l_\tau^*). \qquad \text{(A66)}$$

Differentiating (A66) with respect to a_t and evaluating at steady state yields

$$u_{12}(1-F)\left[1-bL(1-\rho L)^{-1}\right]\frac{\partial c_\tau^*}{\partial a_t} \;=\; -u_{22}(1-F)\frac{\partial l_\tau^*}{\partial a_t}\,. \qquad \text{(A67)}$$

Using (A65) and noting $FL=1$ at steady state,[31] (A67) simplifies to

$$\left[1-\beta(\rho+b)F\right](1-F)\left[1-bL(1-\rho L)^{-1}\right]\frac{\partial c_\tau^*}{\partial a_t} \;=\; 0, \qquad \text{(A68)}$$

which, from (A67), also implies

$$\left[1-\beta(\rho+b)F\right](1-F)\,\frac{\partial l_\tau^*}{\partial a_t} \;=\; 0. \qquad \text{(A69)}$$

Equations (A68) and (A69) hold for all $\tau \geq t$, hence we can invert the $\left[1-\beta(\rho+b)F\right]$ operator forward to get

$$(1-F)\left[1-bL(1-\rho L)^{-1}\right]\frac{\partial c_\tau^*}{\partial a_t} \;=\; 0, \qquad \text{(A70)}$$

$$(1-F)\,\frac{\partial l_\tau^*}{\partial a_t} \;=\; 0. \qquad \text{(A71)}$$

Finally, we can apply $(1-\rho L)$ to both sides of (A70) to get

$$(1-F)\left[1-(\rho+b)L\right]\frac{\partial c_\tau^*}{\partial a_t} \;=\; 0, \qquad \text{(A72)}$$

which then holds for all $\tau \geq t+1$. Thus, whatever the initial responses $\partial c_t^*/\partial a_t$ and $\partial l_t^*/\partial a_t$, we must have:

$$\begin{aligned} E_t\frac{\partial c_{t+1}^*}{\partial a_t} \;&=\; (1+b)\,\frac{\partial c_t^*}{\partial a_t}\,,\\ E_t\frac{\partial c_{t+k}^*}{\partial a_t} \;&=\; \left(1+b(\rho+b)^{k-1}\right)\frac{\partial c_t^*}{\partial a_t}\,, \qquad \text{(A73)}\\ \text{and}\quad E_t\frac{\partial l_{t+k}^*}{\partial a_t} \;&=\; \frac{\partial l_t^*}{\partial a_t}, \quad k=1,2,\ldots \qquad \text{(A74)}\end{aligned}$$

Consumption responds gradually to a surprise change in wealth, while labor moves immediately to its new steady-state level.

From (A73), we can now solve (A65) to get

$$\frac{\partial l_t^*}{\partial a_t} \;=\; -\lambda\,\frac{\partial c_t^*}{\partial a_t}\,. \qquad \text{(A75)}$$

[31] To be precise, $FLx_\tau = E_{\tau-1}x_\tau$, but since the household evaluates these expressions from the perspective of the initial period t, $E_tFLx_\tau = E_tx_\tau$. Formally, take the expectation of (67) at time t and then apply $E_tFL = E_t$ to get (68).

where

$$\lambda \;\equiv\; \frac{w(1-\beta(\rho+b))u_{11} \;+\; (1-\beta\rho)u_{12}}{(1-\beta\rho)u_{22} \;+\; w(1-\beta(\rho+b))u_{12}} \;=\; \frac{u_1 u_{12} - u_2 u_{11}}{u_1 u_{22} - u_2 u_{12}}\,, \tag{A76}$$

where the latter equality follows because $w = -\frac{u_2}{u_1}\frac{1-\beta\rho}{1-\beta(\rho+b)}$ in steady state.

It remains to solve for $\partial c_t^*/\partial a_t$. The household's intertemporal budget constraint implies

$$E_t \sum_{\tau=t}^{\infty} (1+r)^{-(\tau-t)}\, \frac{\partial c_\tau^*}{\partial a_t} \;=\; (1+r) \;+\; w\frac{1+r}{r}\,\frac{\partial l_t^*}{\partial a_t}\,. \tag{A77}$$

Substituting (A73) and (A75) into (A77) and solving for $\partial c_t^*/\partial a_t$ yields

$$\frac{\partial c_t^*}{\partial a_t} \;=\; \frac{(1-\frac{\beta b}{1-\beta\rho})\,r}{1+(1-\frac{\beta b}{1-\beta\rho})w\lambda}\,. \tag{A78}$$

Without habits or labor, an increase in assets would cause consumption to rise by the amount of the income flow from the change in assets—the "golden rule". The presence of habits attenuates this change by the amount $\beta b/(1-\beta\rho)$ in the numerator, and the consumption response is further attenuated by the household's change in labor income, which is accounted for by the denominator of (A78).

Equations (A57), (A58), (A75), and (A78) allow us to compute $R^a(a;h;\theta)$ from Proposition 14:[32]

$$R^a(a;h;\theta) \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\;\frac{(1-\frac{\beta b}{1-\beta\rho})\,r}{1+(1-\frac{\beta b}{1-\beta\rho})w\lambda} \;+\; \alpha\frac{r\,u_1}{u}\left(1-\frac{\beta b}{1-\beta\rho}\right). \tag{A79}$$

Equation (A79) has obvious similarities to the corresponding expressions without habits and with expected utility preferences.

[32] In order to express (A79) in terms of u_1 and u_{11} instead of u_2 and u_{22}, I use $V_1 = (1-\beta(\rho+b))u_1/\big(\beta(1-\beta\rho)\big)$ and differentiate the first-order condition

$$V_1(a_t,h_t;\theta_t) \;=\; (1+r_t)\left(1-\beta bF(1-\beta\rho F)^{-1}\right)u_1(c_\tau^*-h_\tau,l_\tau^*),$$

with respect to a_t to solve for V_{11}.

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