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Risk Aversion and the Labor Margin in Dynamic Equilibrium Models†

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In a static, one-period model with utility $u(\cdot)$ defined over a single consumption good, Arrow (1971) and Pratt (1964) defined the coefficients of absolute and relative risk aversion, $-u''(c)/u'(c)$ and $-cu''(c)/u'(c)$. Difficulties immediately arise, however, when one attempts to generalize these concepts to the case of many periods or many goods (e.g., Kihlstrom and Mirman 1974). These difficulties are particularly pronounced in a dynamic equilibrium model with labor, in which there is a double infinity of goods to consider—consumption and labor in every future period and state of nature—all of which may vary in response to a shock to asset returns or wealth.

The present paper shows how to compute risk aversion in dynamic equilibrium models in general. First, I verify that risk aversion depends on the partial derivatives of the household's value function V with respect to wealth a—that is, the coefficients of absolute and relative risk aversion are essentially $-V_{aa}/V_a$ and $-aV_{aa}/V_a$, respectively. Even though closed-form solutions for the value function do not exist in general, I nevertheless can derive simple, closed-form expressions for risk aversion at the model's nonstochastic steady state, or along a balanced growth path, using the fact that the derivative of the value function with respect to wealth equals the current-period marginal utility of consumption (Benveniste and Scheinkman 1979). Importantly, these closed-form expressions seem to remain very good approximations even far away from the model's steady state.

A main result of the paper is that the household's labor margin has substantial effects on risk aversion, and hence asset prices. Even when labor and consumption are additively separable in utility, they remain connected by the household's budget constraint; in particular, the household can absorb asset return shocks either through changes in consumption, changes in hours worked, or some combination of the two. This ability to absorb shocks along either or both margins greatly alters the household's attitudes toward risk. For example, if the period utility function is given by $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t$, the quantity $-cu_{11}/u_1 = \gamma$ is often referred to as the household's coefficient of relative risk aversion, but in fact the household is *risk*

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neutral with respect to gambles over asset values or wealth. Intuitively, the household is indifferent at the margin between using labor or consumption to absorb a shock to assets, and the household in this example is clearly risk neutral with respect to gambles over hours. More generally, when $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, risk aversion equals $(\gamma^{-1} + \chi^{-1})^{-1}$, a combination of the parameters on the household's consumption and labor margins, reflecting that the household absorbs shocks along both margins.[1]

While modeling risk neutrality is not a main goal of the present paper, risk neutrality nevertheless can be a desirable feature for some applications, such as labor market search or financial frictions, since it allows closed-form solutions to key features of the model.[2] Thus, an additional contribution of the paper is to show ways of modeling risk neutrality that do not require utility to be linear in consumption, which has undesirable implications for interest rates and consumption growth. Instead, linearity of utility in any direction in the (c, l) plane is sufficient.

A final result of the paper is that risk premia computed using the Lucas-Breeden stochastic discounting framework are essentially linear in risk aversion. That is, measuring risk aversion correctly—taking into account the household's labor margin—is necessary for understanding asset prices in the model. Since much recent research has focused on bringing dynamic stochastic general equilibrium (DSGE) models into closer agreement with asset prices,[3] it is surprising that so little attention has been paid to measuring risk aversion correctly in these models. The present paper aims to fill that void.

There are a few previous studies that extend the Arrow-Pratt definition beyond the one-good, one-period case. In a static, multiple-good setting, Stiglitz (1969) measures risk aversion using the household's indirect utility function rather than utility itself, essentially a special case of Proposition 1 of the present paper. Constantinides (1990) measures risk aversion in a dynamic, consumption-only (endowment) economy using the household's value function, another special case of Proposition 1. Boldrin, Christiano, and Fisher (1997) apply Constantinides' definition to some very simple endowment economy models for which they can compute closed-form expressions for the value function, and hence risk aversion. The present paper builds on these studies by deriving closed-form solutions for risk aversion in dynamic equilibrium models in general, demonstrating the importance of the labor margin, and showing the tight link between risk aversion and asset prices in these models.

The remainder of the paper proceeds as follows. Section I describes the dynamic equilibrium framework of the analysis. Section II presents the main ideas of the paper, extending the definition of Arrow-Pratt risk aversion to dynamic equilibrium models with labor and deriving closed-form expressions. Section III demonstrates the close connection between risk aversion and asset pricing. Section IV provides numerical examples that show the accuracy and importance of the closed-form expressions. Section V shows how the expressions apply to the case of balanced growth. Section VI concludes. An Appendix provides details of proofs and computations that are

[1] The intertemporal elasticity of substitution in this example is still $1/\gamma$, so a corollary of this result is that risk aversion and the intertemporal elasticity of substitution are nonreciprocal when labor supply can vary.

[2] See, e.g., Mortensen and Pissarides (1994) and Bernanke, Gertler, and Gilchrist (1999).

[3] See, e.g., Jermann (1998); Tallarini (2000); Boldrin, Christiano, and Fisher (2001); Uhlig (2007); Rudebusch and Swanson (2008, 2009); Van Binsbergen et al. (2010); and Backus, Routledge, and Zin (2008).

outlined in the main text. Results for the case of internal and external habits and Epstein and Zin (1989) preferences are reported in Swanson (2009).

I. Dynamic Equilibrium Framework

A. *The Household's Optimization Problem and Value Function*

Time is discrete and continues forever. At each time t, the household seeks to maximize the expected present discounted value of utility flows,

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, l_\tau), \tag{1}$$

subject to the sequence of asset accumulation equations

$$a_{\tau+1} = (1 + r_\tau) a_\tau + w_\tau l_\tau + d_\tau - c_\tau, \quad \tau = t, \ t+1, \ldots \tag{2}$$

and the no-Ponzi-scheme condition

$$\lim_{T\to\infty} \prod_{\tau=t}^{T} (1 + r_{\tau+1})^{-1} a_{T+1} \geq 0, \tag{3}$$

where E_t denotes the mathematical expectation conditional on the household's information set at time t, $\beta \in (0,1)$ is the household's discount factor, $(c_t, l_t) \in \Omega \subseteq \mathbb{R}^2$ denotes the household's consumption and labor choice in period t, a_t is the household's beginning-of-period assets, and w_t, r_t, and d_t denote the real wage, interest rate, and net transfer payments at time t. There is a finite-dimensional Markovian state vector $\boldsymbol{\theta}_t$ that is exogenous to the household and governs the processes for w_t, r_t, and d_t. Conditional on $\boldsymbol{\theta}_t$, the household knows the time-t values for w_t, r_t, and d_t. The state vector and information set of the household's optimization problem at each date t is thus $(a_t; \boldsymbol{\theta}_t)$. Let X denote the domain of $(a_t; \boldsymbol{\theta}_t)$, and let $\Gamma : X \to \Omega$ describe the set-valued correspondence of feasible choices for (c_t, l_t) for each given $(a_t; \boldsymbol{\theta}_t)$.

We make the following regularity assumptions regarding the period utility function u:

ASSUMPTION 1: *The function $u : \Omega \to \mathbb{R}$ is increasing in its first argument, decreasing in its second, twice-differentiable, and strictly concave.*

In addition to Assumption 1, a few more technical conditions are required to ensure the value function for the household's optimization problem exists and satisfies the Bellman equation (Stokey, Lucas, and Prescott 1990; Alvarez and Stokey 1998; and Rincón-Zapatera and Rodríguez-Palmero 2003 give different sets of such sufficient conditions). The details of these conditions are tangential to the present paper, so we simply assume:

ASSUMPTION 2: *The value function $V : X \to \mathbb{R}$ for the household's optimization problem exists and satisfies the Bellman equation*

$$V(a_t; \boldsymbol{\theta}_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \boldsymbol{\theta}_t)} u(c_t, l_t) + \beta E_t V(a_{t+1}; \boldsymbol{\theta}_{t+1}), \tag{4}$$

where a_{t+1} *is given by* (2).

The same technical conditions, plus Assumption 1, guarantee the existence of a unique optimal choice for (c_t, l_t) at each point in time, given $(a_t; \theta_t)$. Let $c_t^* \equiv c^*(a_t; \theta_t)$ and $l_t^* \equiv l^*(a_t; \theta_t)$ denote the household's optimal choices of c_t and l_t as functions of the state $(a_t; \theta_t)$. Then V can be written as

$$V(a_t; \theta_t) = u(c_t^*, l_t^*) + \beta E_t V(a_{t+1}^*; \theta_{t+1}), \tag{5}$$

where $a_{t+1}^* \equiv (1 + r_t)a_t + w_t l_t^* + d_t - c_t^*$. We also assume that these solutions are interior:

ASSUMPTION 3: *For any* $(a_t; \theta_t) \in X$, *the household's optimal choice* (c_t^*, l_t^*) *exists, is unique, and lies in the interior of* $\Gamma(a_t; \theta_t)$.

Intuitively, Assumption 3 requires the partial derivatives of u to grow sufficiently large toward the boundary that only interior solutions for c_t^* and l_t^* are optimal for all $(a_t; \theta_t) \in X$.

Assumptions 1–3 guarantee V is continuously differentiable with respect to a and satisfies the Benveniste-Scheinkman equation, but we will require slightly more than this below:

ASSUMPTION 4: *For any* $(a_t; \theta_t)$ *in the interior of* X, *the second derivative of* V *with respect to its first argument,* $V_{11}(a_t; \theta_t)$, *exists.*

Assumption 4 also implies differentiability of the optimal policy functions, c^* and l^*, with respect to a. Santos (1991) provides relatively mild sufficient conditions for Assumption 4 to be satisfied; intuitively, u must be strongly concave.

B. *Representative Household and Steady-State Assumptions*

Up to this point, we have considered the case of a single household in isolation, leaving the other households of the model and the production side of the economy unspecified. Implicitly, the other households and production sector jointly determine the process for θ_t (and hence w_t, r_t, and d_t), and much of the analysis below does not need to be any more specific about these processes than this. To move from general expressions for risk aversion to more concrete, closed-form expressions, however, we adopt three standard assumptions from the DSGE literature:[4]

ASSUMPTION 5: *The household is* infinitesimal.

ASSUMPTION 6: *The household is* representative.

[4] Alternative assumptions about the nature of the other households in the model or the production sector may also allow for closed-form expressions for risk aversion. The assumptions used here are standard, however, and thus the most natural to pursue.

ASSUMPTION 7: *The model has a* nonstochastic steady state, $x_t = x_{t+k}$ *for* $k = 1, 2, \ldots,$ *and* $x \in \{c, l, a, w, r, d, \theta\}$.

Assumption 5 implies that an individual household's choices for c_t and l_t have no effect on the aggregate quantities w_t, r_t, d_t, and θ_t. Assumption 6 implies that, when the economy is at the nonstochastic steady state, any individual household finds it optimal to choose the steady-state values of c and l given a and θ. Throughout the text, we drop the subscript t on a variable to denote its steady-state value.

It is important to note that Assumptions 6 and 7 do not prohibit us from offering an individual household a hypothetical gamble of the type described below. The steady state of the model serves only as a reference point around which the *aggregate* variables w, r, d, and θ and the *other households'* choices of c, l, and a can be predicted with certainty. This reference point is important because it is there that we can compute closed-form expressions for risk aversion.

Finally, many dynamic models do not have a steady state per se, but rather a balanced growth path. The results below carry through essentially unchanged to the case of balanced growth. For ease of exposition, we restrict attention in Sections II–IV to a steady state, and show in Section V the adjustments required under the more general

ASSUMPTION 7′: *The model has a* balanced growth path *that can be renormalized to a nonstochastic steady state after a suitable change of variables.*

II. Risk Aversion

A. *The Coefficient of Absolute Risk Aversion*

The household's risk aversion at time t generally depends on the household's state vector at time t, $(a_t; \theta_t)$. Given this state, we consider the household's aversion to a hypothetical one-shot gamble in period t of the form

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + \sigma\varepsilon_{t+1}, \tag{6}$$

where ε_{t+1} is a random variable representing the gamble, with bounded support $[\underline{\varepsilon}, \overline{\varepsilon}]$, mean zero, unit variance, independent of θ_τ for all τ, and independent of a_τ, c_τ, and l_τ for all $\tau \leq t$. A few words about equation (6) are in order: First, the gamble is dated $t + 1$ to clarify that its outcome is not in the household's information set at time t. Second, c_t cannot be made the subject of the gamble without substantial modifications to the household's optimization problem, because c_t is a choice variable under control of the household at time t. Equation (6) is clearly equivalent to a one-shot gamble over net transfers d_t or asset returns r_t, however, both of which are exogenous to the household. Indeed, thinking of the gamble as being over r_t helps to illuminate the connection between equation (6) and the price of risky assets, to which we will return in Section III. As shown there, the household's aversion to the gamble in equation (6) is directly linked to the premium households require to hold risky assets.

Following Arrow (1971) and Pratt (1964), we can ask what one-time fee μ the household would be willing to pay in period t to avoid the gamble in equation (6):

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu. \tag{7}$$

The quantity μ that makes the household just indifferent between equations (6) and (7), for infinitesimal σ and μ, is the household's coefficient of absolute risk aversion, which we denote by R^a.[5] Formally, this corresponds to the following definition:

DEFINITION 1: *Let $(a_t; \theta_t)$ be an interior point of X, let $\tilde{V}(a_t; \theta_t; \sigma)$ denote the value function for the household's optimization problem inclusive of the one-shot gamble* (6), *and let $\mu(a_t; \theta_t; \sigma)$ denote the value of μ that satisfies $V(a_t - \frac{\mu}{1+r_t}; \theta_t) = \tilde{V}(a_t; \theta_t; \sigma)$. The household's* coefficient of absolute risk aversion at $(a_t; \theta_t)$, *denoted $R^a(a_t; \theta_t)$, is given by $R^a(a_t; \theta_t) = \lim_{\sigma \to 0} \frac{\mu(a_t; \theta_t; \sigma)}{\sigma^2/2}$.*

In Definition 1, $\mu(a_t; \theta_t; \sigma)$ denotes the household's "willingness to pay" to avoid a one-shot gamble of size σ in equation (6). As in Arrow (1971) and Pratt (1964), R^a denotes the limit of the household's willingness to pay per unit of variance as this variance becomes small. Note that $R^a(a_t; \theta_t)$ depends on the economic state because $\mu(a_t; \theta_t; \sigma)$ depends on that state. Proposition 1 shows that $\tilde{V}(a_t; \theta_t; \sigma)$, $\mu(a_t; \theta_t; \sigma)$, and $R^a(a_t; \theta_t)$ in Definition 1 are well defined and that $R^a(a_t; \theta_t)$ equals the "folk wisdom" value of $-V_{11}/V_1$:[6]

PROPOSITION 1: *Let $(a_t; \theta_t)$ be an interior point of X. Given Assumptions* 1–5, *$\tilde{V}(a_t; \theta_t; \sigma)$, $\mu(a_t; \theta_t; \sigma)$, and $R^a(a_t; \theta_t)$ exist and*

$$R^a(a_t; \theta_t) = \frac{-E_t V_{11}(a^*_{t+1}; \theta_{t+1})}{E_t V_1(a^*_{t+1}; \theta_{t+1})}, \tag{8}$$

where V_1 and V_{11} denote the first and second partial derivatives of V with respect to its first argument. Given Assumptions 6–7, *equation* (8) *can be evaluated at the steady state to yield*

$$R^a(a; \theta) = \frac{-V_{11}(a; \theta)}{V_1(a; \theta)}. \tag{9}$$

PROOF:

See Appendix.

Equations (8) and (9) are essentially the definition of risk aversion in Constantinides (1990), and have obvious similarities to Arrow (1971) and Pratt (1964). Here, of

[5] We defer discussion of relative risk aversion until the next subsection because defining total household wealth is complicated by the presence of human capital—that is, the household's labor income.

[6] See, e.g., Constantinides (1990); Farmer (1990); Cochrane (2001); and Flavin and Nakagawa (2008). For the more general case of Epstein-Zin (1989) preferences, equation (8) no longer holds and there is no folk wisdom; see Swanson (2009) for the more general formulae corresponding to that case.

course, it is the curvature of the value function V with respect to assets that matters, rather than the curvature of the period utility function u with respect to consumption.[7]

Deriving the coefficient of absolute risk aversion in Proposition 1 is simple enough, but the problem with equations (8) and (9) is that closed-form expressions for V do not exist in general, even for the simplest dynamic models with labor. This difficulty may help to explain the popularity of "shortcut" approaches to measuring risk aversion, notably $-u_{11}(c_t^*, l_t^*)/u_1(c_t^*, l_t^*)$, which has no clear relationship to equations (8)–(9) except in the one-good one-period case. Boldrin, Christiano, and Fisher (1997) derive closed-form solutions for V—and hence risk aversion—for some very simple, consumption-only endowment economy models, but this approach is a non-starter for even the simplest dynamic models with labor.

We solve this problem by observing that V_1 and V_{11} often can be computed even when closed-form solutions for V cannot be. For example, the Benveniste-Scheinkman equation

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c_t^*, l_t^*) \tag{10}$$

states that the marginal value of a dollar of assets equals the marginal utility of consumption times $1 + r_t$ (the interest rate appears here because beginning-of-period assets in the model generate income in period t). In equation (10), u_1 is a known function. Although closed-form solutions for the functions c^* and l^* are not known in general, the points c_t^* and l_t^* often are known—for example, when they are evaluated at the nonstochastic steady state, c and l. Thus, we can compute V_1 at the nonstochastic steady state by evaluating equation (10) at that point.

We compute V_{11} by noting that equation (10) holds for general a_t; hence it can be differentiated to yield

$$V_{11}(a_t; \theta_t) = (1 + r_t) \left[u_{11}(c_t^*, l_t^*) \frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*, l_t^*) \frac{\partial l_t^*}{\partial a_t} \right]. \tag{11}$$

All that remains is to find the derivatives $\partial c_t^* / \partial a_t$ and $\partial l_t^* / \partial a_t$.

We solve for $\partial l_t^* / \partial a_t$ by differentiating the household's intratemporal optimality condition:

$$-u_2(c_t^*, l_t^*) = w_t u_1(c_t^*, l_t^*), \tag{12}$$

with respect to a_t, and rearranging terms to yield

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda_t \frac{\partial c_t^*}{\partial a_t}, \tag{13}$$

[7] Arrow (1971) and Pratt (1964) occasionally refer to utility as being defined over "money," so one could argue that they always intended for risk aversion to be measured using indirect utility or the value function.

where

$$\lambda_t \equiv \frac{w_t u_{11}(c_t^*, l_t^*) + u_{12}(c_t^*, l_t^*)}{u_{22}(c_t^*, l_t^*) + w_t u_{12}(c_t^*, l_t^*)} = \frac{u_1(c_t^*, l_t^*) u_{12}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*) u_{11}(c_t^*, l_t^*)}{u_1(c_t^*, l_t^*) u_{22}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*) u_{12}(c_t^*, l_t^*)}. \tag{14}$$

If consumption and leisure in period t are normal goods, then $\lambda_t > 0$, although we do not require this restriction below. It now only remains to solve for the derivative $\partial c_t^* / \partial a_t$.

Intuitively, $\partial c_t^* / \partial a_t$ should not be too difficult to compute: it is just the household's marginal propensity to consume today out of a change in assets, which we can deduce from the household's Euler equation and budget constraint. Differentiating the Euler equation:

$$u_1(c_t^*, l_t^*) = \beta E_t (1 + r_{t+1}) u_1(c_{t+1}^*, l_{t+1}^*), \tag{15}$$

with respect to a_t yields[8]

$$u_{11}(c_t^*, l_t^*) \frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*, l_t^*) \frac{\partial l_t^*}{\partial a_t} = \beta E_t (1 + r_{t+1}) \left[u_{11}(c_{t+1}^*, l_{t+1}^*) \frac{\partial c_{t+1}^*}{\partial a_t} + u_{12}(c_{t+1}^*, l_{t+1}^*) \frac{\partial l_{t+1}^*}{\partial a_t} \right]. \tag{16}$$

Substituting in for $\partial l_t^* / \partial a_t$ gives

$$\left(u_{11}(c_t^*, l_t^*) - \lambda_t u_{12}(c_t^*, l_t^*)\right) \frac{\partial c_t^*}{\partial a_t} = \beta E_t (1 + r_{t+1}) \left(u_{11}(c_{t+1}^*, l_{t+1}^*) - \lambda_{t+1} u_{12}(c_{t+1}^*, l_{t+1}^*)\right) \frac{\partial c_{t+1}^*}{\partial a_t}. \tag{17}$$

Evaluating equation (17) at steady state, $\beta = (1 + r)^{-1}$, $\lambda_t = \lambda_{t+1} = \lambda$, and the u_{ij} cancel, giving

$$\frac{\partial c_t^*}{\partial a_t} = E_t \frac{\partial c_{t+1}^*}{\partial a_t} = E_t \frac{\partial c_{t+k}^*}{\partial a_t}, \quad k = 1, 2, \ldots \tag{18}$$

$$\frac{\partial l_t^*}{\partial a_t} = E_t \frac{\partial l_{t+1}^*}{\partial a_t} = E_t \frac{\partial l_{t+k}^*}{\partial a_t}, \quad k = 1, 2, \ldots \tag{19}$$

[8] By $\partial c_{t+1}^* / \partial a_t$ I mean $\frac{\partial c_{t+1}^*}{\partial a_{t+1}} \frac{da_{t+1}^*}{da_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}} \left[1 + r_{t+1} + w_t \frac{\partial l_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t} \right]$, and analogously for $\partial l_{t+1}^* / \partial a_t$, $\partial c_{t+2}^* / \partial a_t$, $\partial l_{t+2}^* / \partial a_t$, etc.

In other words, whatever the change in the household's consumption today, it must be the same as the expected change in consumption tomorrow, and the expected change in consumption at each future date $t + k$.[9]

The household's budget constraint is implied by asset accumulation equation (2) and no-Ponzi condition (3). Differentiating equation (2) with respect to a_t, evaluating at steady state, and applying equations (3), (18), and (19) gives

$$\frac{1 + r}{r} \frac{\partial c_t^*}{\partial a_t} = (1 + r) + \frac{1 + r}{r} w \frac{\partial l_t^*}{\partial a_t}. \tag{20}$$

That is, the expected present value of changes in household consumption must equal the change in assets (times $1 + r$) plus the expected present value of changes in labor income.

Combining equation (20) with equation (13), we can solve for $\partial c_t^* / \partial a_t$ evaluated at the steady state:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1 + w\lambda}. \tag{21}$$

In response to a unit increase in assets, the household raises consumption in every period by the extra asset income, r (the "golden rule"), adjusted downward by the amount $1 + w\lambda$ that takes into account the household's decrease in labor income.

We can now compute the household's coefficient of absolute risk aversion. Substituting equations (10), (11), (13), (14), and (21) into equation (9), we have proved:[10]

PROPOSITION 2: *Given Assumptions* 1–7, *the household's coefficient of absolute risk aversion* $R^a(a_t; \theta_t)$ *evaluated at steady state satisfies*

$$R^a(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda}, \tag{22}$$

where u_1, u_{11}, *and* u_{12} *denote the corresponding partial derivatives of u evaluated at the steady state* (c, l), *and* λ *is given by equation* (14) *evaluated at steady state.*

When there is no labor margin in the model, Proposition 2 has the following corollary:

COROLLARY 1: *Given Assumptions* 1–7, *suppose that* l_t *is fixed exogenously at some* $\bar{l} \in \mathbb{R}$ *for all t and that the household chooses* c_t *optimally at each t given this constraint. Then*

$$R^a(a; \theta) = \frac{-u_{11}}{u_1} r. \tag{23}$$

[9] Note that this equality does not follow from the steady-state assumption. For example, in a model with internal habits, considered in Swanson (2009), the individual household's optimal consumption response to a change in assets increases with time, even starting from steady state.

[10] Equations (10), (11), (13), (14), and (21) are also valid for the more general case of Epstein-Zin preferences, although equation (9) is not. See Swanson (2009) for the expression corresponding to equation (22) in that case.

PROOF:

The assumptions and steps leading up to Proposition 2, adjusted to the dynamic consumption-only case, are essentially the same as the above with $\lambda_t = 0$.

Proposition 2 and Corollary 1 are remarkable. First, the household's coefficient of absolute risk aversion in equation (23) is just the traditional measure, $-u_{11}/u_1$, times r, which translates assets into current-period consumption.[11] In other words, for *any* period utility function u, the traditional, static measure of risk aversion is also the correct measure in the dynamic context, regardless of whether or not u is homothetic or the rest of the model is homogeneous, whether or not we can solve for V, and no matter what the functional forms of u and V.

More generally, when households have a labor margin, Proposition 2 shows that risk aversion is less than the traditional measure by the factor $1 + w\lambda$, *even when consumption and labor are additively separable* (i.e., $u_{12} = 0$). Even in the additively separable case, households can partially absorb shocks to income through changes in hours worked. As a result, c_t^* depends on household labor supply, so labor and consumption are indirectly connected through the budget constraint.[12] When $u_{12} \neq 0$, risk aversion in Proposition 2 is further attenuated or amplified by the direct interaction between consumption and labor in utility, u_{12}. Note, however, that regardless of the signs of λ and u_{12}, risk aversion is always positive and always reduced, on net, when households can vary their labor supply:

COROLLARY 2: *The coefficient of absolute risk aversion* (22) *is positive and less than or equal to* (23),

$$\frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} \leq \frac{-u_{11}}{u_1} r. \tag{24}$$

If $r < 1$, *then equation* (22) *is also less than* $-u_{11}/u_1$.

PROOF:

Substituting in for λ and w, equation (22) can be written as

$$\frac{-r u_{11}}{u_1} \frac{u_{11} u_{22} - u_{12}^2}{u_{11} u_{22} - 2 \frac{u_2}{u_1} u_{11} u_{12} + \left(\frac{u_2}{u_1}\right)^2 u_{11}^2} \tag{25}$$

$$= -\frac{r u_{11}}{u_1} \frac{1}{1 + \dfrac{\left(\frac{u_2}{u_1} u_{11} - u_{12}\right)^2}{u_{11} u_{22} - u_{12}^2}}.$$

[11] A gamble over a lump sum of \$$X$ is equivalent here to a gamble over an annuity of \$$X/r$. Thus, even though V_{11}/V_1 is different from u_{11}/u_1 by a factor of r, this difference is exactly the same as a change from lump-sum to annuity units. Thus, the difference in scale is essentially one of units.

[12] Uhlig (2007) notes that, if households have Epstein-Zin preferences, then leisure must be taken into account in pricing assets because the value function V appears in the stochastic discount factor, and V depends on leisure. The present paper makes the point that the labor margin affects asset prices even in the case of additively separable expected utility preferences, because the labor margin changes the household's consumption process. The present paper also derives closed-form expressions, relates them to asset prices, and shows that those expressions remain good approximations away from the steady state.

Strict concavity of u implies that $u_{11} < 0$ and $u_{11}u_{22} - u_{12}^2 > 0$, hence the right-hand side of equation (25) is positive and less than or equal to $-ru_{11}/u_1$.

Since r denotes the net interest rate, $r \ll 1$ in typical calibrations, satisfying the condition at the end of Corollary 2.

The household's labor margin can have dramatic effects on risk aversion. For example, from the left-hand side of equation (25) it is apparent that, no matter how large is $-u_{11}/u_1$, risk aversion can be arbitrarily small as the discriminant, $u_{11}u_{22} - u_{12}^2$, approaches zero.[13] In other words, risk aversion depends on the concavity of u in all dimensions rather than in just one dimension. Even when u_{11} is very large, the household can still be risk neutral if u_{22} is small or the cross-effect u_{12} is sufficiently large. Geometrically, if there exists any direction in (c, l)-space along which u has zero curvature, the household will optimally choose to absorb shocks to income along that line, resulting in risk-neutral behavior.

We provide some more concrete examples of risk aversion calculations in Section IIC, below, after first defining relative risk aversion.

B. *The Coefficient of Relative Risk Aversion*

The difference between absolute and relative risk aversion is the size of the hypothetical gamble faced by the household. If the household faces a one-shot gamble of size A_t in period t:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + A_t \sigma \varepsilon_{t+1}, \tag{26}$$

or the household can pay a one-time fee $A_t \mu$ in period t to avoid this gamble, then it follows from Proposition 1 that $\lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2$ for this gamble is given by

$$\frac{-A_t E_t V_{11}(a_{t+1}^*; \boldsymbol{\theta}_{t+1})}{E_t V_1(a_{t+1}^*; \boldsymbol{\theta}_{t+1})}. \tag{27}$$

The natural definition of A_t, considered by Arrow (1971) and Pratt (1964), is the household's wealth at time t. The gamble in (26) is then over a fraction of the household's wealth and (27) is referred to as the household's coefficient of relative risk aversion.

In models with labor, however, household wealth can be more difficult to define because of the presence of human capital. In these models, there are two natural definitions of human capital, so we consequently define two measures of household wealth A_t and two coefficients of relative risk aversion (27).

First, when the household's time endowment is not well defined—as when $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}$ and no upper bound $\bar{l}$ on l_t is specified, or $\bar{l}$ is specified but is completely arbitrary—it is most natural to define human capital as the present discounted value of labor income, $w_t l_t^*$. Equivalently, total household wealth

[13] Note that the denominator of equation (25), $u_{11}u_{22} - 2(u_2/u_1)u_{11}u_{12} + (u_2/u_1)^2 u_{11}^2$, must also not vanish, which will be true so long as $[-u_2, u_1]'$ is not in the nullspace of the Hessian of u.

A_t equals the present discounted value of consumption, which follows from the budget constraint (2)–(3). We state this formally as

DEFINITION 2: *The* consumption-only coefficient of relative risk aversion, *denoted* $R^c(a_t; \theta_t)$, *is given by equation* (27) *with* $A_t \equiv (1 + r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} c_\tau^*$, *the present discounted value of household consumption, where* $m_{t,\tau}$ *denotes the stochastic discount factor* $\beta^{\tau-t} u_1(c_\tau^*, l_\tau^*)/u_1(c_t^*, l_t^*)$.

The factor $(1 + r_t)^{-1}$ in the definition expresses wealth A_t in beginning- rather than end-of-period-t units, so that in steady state $A = c/r$ and $R^c(a; \theta)$ is given by

$$R^c(a; \theta) = \frac{-AV_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda}. \tag{28}$$

Note that Corollary 2 implies $R^c(a; \theta) \leq -cu_{11}/u_1$.

Alternatively, when the household's time endowment $\bar{l}$ is well specified, we can define human capital to be the present discounted value of the household's time endowment, $w_t \bar{l}$. In this case, total household wealth $\tilde{A}_t$ equals the present discounted value of leisure $w_t(\bar{l} - l_t^*)$ plus consumption c_t^*, from (2)–(3). We thus have

DEFINITION 3: *The* consumption-and-leisure coefficient of relative risk aversion, *denoted* $R^{cl}(a_t; \theta_t)$, *is given by equation* (27) *with* $\tilde{A}_t \equiv (1 + r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau}(c_\tau^* + w_\tau(\bar{l} - l_\tau^*))$.

In steady state, $\tilde{A} = (c + w(\bar{l} - l))/r$, and $R^{cl}(a; \theta)$ is given by

$$R^{cl}(a; \theta) = \frac{-\tilde{A}V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1 + w\lambda}. \tag{29}$$

Of course, (28) and (29) are related by the ratio of the two gambles, $(c + w(\bar{l} - l))/c$. Note $R^{cl}(a; \theta)$ may be greater or less than $-cu_{11}/u_1$, depending on the size of $w(\bar{l} - l)$ relative to c.

Other definitions of relative risk aversion, corresponding to alternative definitions of wealth, are also possible, but Definitions 2–3 are the most natural for several reasons. First, both definitions reduce to the usual present discounted value of income or consumption when there is no human capital in the model. Second, both measures reduce to the traditional $-cu_{11}/u_1$ when there is no labor margin in the model—that is, when $\lambda = 0$. Third, in steady state the household consumes exactly the flow of income from its wealth, rA, consistent with standard permanent income theory (where one must include the value of leisure $w(\bar{l} - l)$ as part of consumption when the value of leisure is included in wealth).

Finally, note that neither measure of relative risk aversion is reciprocal to the intertemporal elasticity of substitution:

COROLLARY 3: *Given Assumptions* 1–7, (*i*) $R^c(a; \theta)$ *and the intertemporal elasticity of substitution are reciprocal if and only if* $\lambda = 0$; (*ii*) $R^{cl}(a; \theta)$ *and the intertemporal elasticity of substitution are reciprocal if and only if* $\lambda = (\bar{l} - l)/c$.

PROOF:

The case $w = 0$ is ruled out by Assumptions 1 and 3. The intertemporal elasticity of substitution, evaluated at steady state, is given by $(dc_{t+1}^* - dc_t^*)/(cd\log(1 + r_{t+1}))$, which equals $-u_1/(c(u_{11} - \lambda u_{12}))$ by a calculation along the lines of equation (17), holding w_t fixed but allowing l_t^* and l_{t+1}^* to vary endogenously. The corollary then follows by comparison to equations (28) and (29).

C. *Examples*

EXAMPLE 1: Consider the King, Plosser, and Rebelo-type (1988) period utility function:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}(1 - l_t)^{\chi(1-\gamma)}}{1-\gamma}, \tag{30}$$

where $\gamma > 0$, $\gamma \neq 1$, $\chi > 0$, $\bar{l} = 1$, and $\chi(1-\gamma) < \gamma$ for concavity. The traditional measure of risk aversion for (30) is $-cu_{11}/u_1 = \gamma$, but the labor margin implies

$$R^{cl}(a;\boldsymbol{\theta}) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(1 - l)}{1 + w\lambda} = \gamma - \chi(1 - \gamma). \tag{31}$$

Thus, $R^{cl}(a;\boldsymbol{\theta})$ depends on both γ and χ, the coefficients on the consumption and labor margins. Note that concavity of (30) implies (31) is positive. Neither (31) nor

$$R^{c}(a;\boldsymbol{\theta}) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda} = \frac{\gamma - \chi(1 - \gamma)}{1 + \chi} \tag{32}$$

equals the traditional measure γ, except for the special case $\chi = 0$. As χ approaches $\gamma/(1-\gamma)$—that is, as utility approaches Cobb-Douglas—the household becomes risk neutral; in this case, utility along the line $c_t = w_t(1 - l_t)$ is linear, so the household finds it optimal to absorb shocks to income or assets along that line.

Also note that if

$$u(c_t, l_t) = \frac{(c_t^{1-\chi}(1 - l_t)^{\chi})^{1-\gamma}}{1 - \gamma}, \tag{33}$$

where $\chi \in (0,1)$, then $R^{cl}(a;\boldsymbol{\theta}) = \gamma$, the same as regarding consumption and leisure as a single, composite good.

EXAMPLE 2: *Consider the additively separable period utility function:*

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta\frac{l_t^{1+\chi}}{1+\chi}, \tag{34}$$

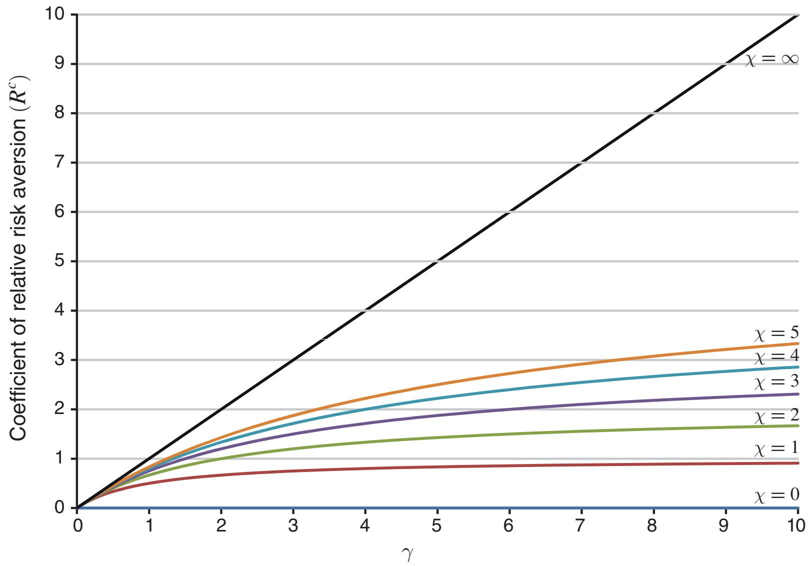

Figure 1

Notes: Coefficient of relative risk aversion $R^c(a;\boldsymbol{\theta})$ for period utility function $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ in Example 2, as a function of the traditional measure γ, for different values of χ. See text for details.

where γ, χ, $\eta > 0$. The traditional measure of risk aversion for these preferences is γ, but

$$R^c(a;\boldsymbol{\theta}) = \frac{-cu_{11}}{u_1} \frac{1}{1 + w\frac{wu_{11}}{u_{22}}} = \frac{\gamma}{1 + \frac{\gamma}{\chi}\frac{wl}{c}}. \tag{35}$$

$R^{cl}(a_t;\boldsymbol{\theta}_t)$ is not well defined in this example (it can be made arbitrarily large or small just by varying the household's time endowment $\bar{l}$), so I consider only R^c.

To simplify the following discussion, restrict attention further in this example to the special case $c \approx wl$,[14] an assumption that is made *in this paragraph only* and nowhere else in the paper. In this special case, the closed-form expression in equation (35) can be simplified further to

$$R^c(a;\boldsymbol{\theta}) \approx \frac{1}{\frac{1}{\gamma} + \frac{1}{\chi}}. \tag{36}$$

Note that equation (36) is less than the traditional measure of risk aversion by a factor of $1 + \gamma/\chi$. Thus, if $\gamma = 2$ and $\chi = 1$—parameter values that are well within the range of estimates in the literature—then the household's true risk aversion is less than the traditional measure by a factor of about three. This point is illustrated in Figure 1, which graphs the coefficient of relative risk aversion for this example as a function of the traditional measure, γ, for several different values of χ. If χ is

[14] In steady state, $c = ra + wl + d$, so $c = wl$ holds exactly if there is neither capital nor transfers in the model. In any case, $ra + d$ is typically small, since $r \approx 0.01$.

very large, then the bias from using the traditional measure is small because household labor supply is essentially fixed.[15] As χ approaches 0, however, a common benchmark in the literature, the bias explodes and true risk aversion approaches zero—the household becomes risk neutral. Intuitively, households with linear disutility of work are risk neutral with respect to gambles over income or assets because they can completely offset those gambles at the margin by working more or fewer hours, and households with linear disutility of work are clearly risk neutral with respect to gambles over hours.

III. Risk Aversion and Asset Pricing

The preceding sections show that the labor margin has important implications for risk aversion with respect to gambles over income or wealth. I now show that risk aversion with respect to these gambles is the right concept for asset pricing.

A. *Measuring Risk Aversion with V as Opposed to u*

Some comparison of the expressions $-V_{11}/V_1$ and $-u_{11}/u_1$ helps to clarify why the former measure is the relevant one for pricing assets, such as stocks or bonds. From Proposition 1, $-V_{11}/V_1$ is the Arrow-Pratt coefficient of absolute risk aversion for gambles over income or wealth in period t. In contrast, the expression $-u_{11}/u_1$ is the risk aversion coefficient for a hypothetical gamble in which the household is *forced to consume immediately* the outcome of the gamble. Clearly, it is the former concept that corresponds to the stochastic payoffs of a standard asset such as a stock or bond. In order for $-u_{11}/u_1$ to be the relevant measure for pricing a security, it is not enough that the security pay off in units of consumption in period $t+1$. The household would additionally have to be prevented from adjusting its consumption and labor choices in period $t+1$ in response to the security's payoffs, so that the household is forced to absorb those payoffs into period $t+1$ consumption. It is difficult to imagine such a security—all standard securities in financial markets correspond to gambles over income, assets, or wealth, for which the $-V_{11}/V_1$ measure of risk aversion is the appropriate one.[16]

B. *Risk Aversion, the Stochastic Discount Factor, and Risk Premia*

Let $m_{t+1} = \beta u_1(c^*_{t+1}, l^*_{t+1})/u_1(c^*_t, l^*_t)$ denote the household's stochastic discount factor and let p_t denote the cum-dividend price of a risky asset at time t, with $E_t p_{t+1}$

[15] Similarly, if γ is very small, the bias from using the traditional measure is small because the household chooses to absorb income shocks almost entirely along its consumption margin. As a result, the labor margin is again almost inoperative.

[16] Here and throughout the paper, we take it as given that the gambles of interest are those that occur most frequently in the literature: namely, gambles over income, wealth, or asset returns (either real or nominal), for which Definitions 1–3 are the "correct" or "appropriate" measures of risk aversion. The reader should bear in mind, however, that for other gambles—such as one that the household is forced to absorb entirely in current-period consumption—alternative measures of risk aversion such as the traditional $-u_{11}/u_1$ may be appropriate instead. Thus, the terms "correct" or "appropriate" in the present paper should be thought of as having the qualifier "for gambles over income, wealth, or asset returns."

normalized to unity. Define the risk premium on the asset to be the percentage difference between the risk-neutral price of the asset and its actual price:

$$(37)\quad (E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1})/E_t m_{t+1} = -\mathrm{Cov}_t(dm_{t+1}, dp_{t+1})/E_t m_{t+1},$$

where Cov_t denotes the covariance conditional on information at time t, and $dx_{t+1} \equiv x_{t+1} - E_t x_{t+1}, x \in \{m, p\}$. For small changes dc^*_{t+1} and dl^*_{t+1}, we have, to first order:

$$(38)\qquad dm_{t+1} = \frac{\beta}{u_1(c^*_t, l^*_t)}\left[u_{11}(c^*_{t+1}, l^*_{t+1})dc^*_{t+1} + u_{12}(c^*_{t+1}, l^*_{t+1})dl^*_{t+1}\right].$$

In equation (38), the household's labor margin affects m_{t+1} and hence asset prices for two reasons: First, if $u_{12} \neq 0$, changes in l_{t+1} directly affect the household's marginal utility of consumption. Second, even if $u_{12} = 0$, the presence of the labor margin affects how the household responds to shocks and hence affects dc^*_{t+1}.

Intuitively, one can already see the relationship between risk aversion and dm_{t+1} in equation (38): if $dl^*_{t+1} = -\lambda dc^*_{t+1}$ and $dc^*_{t+1} = r da_{t+1}/(1 + w\lambda)$, as in Section II, then $dm_{t+1} = R^a(a; \boldsymbol{\theta}) da_{t+1}$. In actuality, the relationship is more complicated than this because $\boldsymbol{\theta}$ (and hence w, r, and d) may change as well as a in response to macroeconomic shocks. For example, differentiating equation (12) and evaluating at steady state implies

$$(39)\qquad dl^*_{t+1} = -\lambda dc^*_{t+1} - \frac{u_1}{u_{22} + w u_{12}} dw_{t+1}$$

to first order. The expression for dc^*_{t+1} is somewhat more complicated:

LEMMA 1: *Given Assumptions* 1–7,

$$(40)\quad dc^*_{t+1} = \frac{r}{1 + w\lambda}\left[da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1 + r)^k}(l dw_{t+k} + dd_{t+k} + a dr_{t+k})\right]$$
$$+ \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2} dw_{t+1}$$
$$+ \frac{-u_1}{u_{11} - \lambda u_{12}} E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1 + r)^k}\left(\frac{r\lambda}{1 + w\lambda} dw_{t+k} - \beta\, dr_{t+k+1}\right)$$

to first order, *evaluated at the steady state.*

PROOF:

The expression follows from (2), (3), and (15). See the Appendix for details.

For the Arrow-Pratt one-shot gamble considered in Section II, the aggregate variables w, r, and d were held constant, so equations (39)–(40) reduced to equations (13) and (21). The term in square brackets in equation (40) describes the change in the present value of household income, and thus the first line of (40) describes the income effect on consumption. The last two lines of (40) describes the substitution effect: changes in consumption due to changes in current and future wages and interest rates. (Recall $-u_1/(c(u_{11} - \lambda u_{12}))$ is the intertemporal elasticity of substitution.)

We are now in a position to relate risk aversion to asset prices and risk premia:

PROPOSITION 3: *Given Assumptions* 1–7, *the household's stochastic discount factor satisfies*

$$(41)\quad dm_{t+1} = -\beta R^a(a;\theta)\left[da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}(ldw_{t+k} + dd_{t+k} + adr_{t+k})\right] - \beta E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(\frac{r\lambda}{1+w\lambda}dw_{t+k} - \beta dr_{t+k+1}\right)$$

to first order, *evaluated at steady state. The risk premium in equation* (37) *is given to second order around the steady state by*

$$(42)\qquad R^a(a;\theta)\cdot \mathrm{Cov}_t(dp_{t+1}, d\hat{A}_{t+1}) + \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}),$$

where $d\hat{A}_{t+1}$ denotes the change in wealth given by the quantity in square brackets in equation (41) *and $d\Psi_{t+1}$ denotes the change in wages and interest rates given by the second line of* (41).

PROOF:

Substituting equations (39)–(40) into (38) yields (41). Substituting (41) into (37) yields (42). Note that $\beta = E_t m_{t+1}$. Finally, $\mathrm{Cov}(dx, dy)$ is accurate to second order when dx and dy are accurate to first order.

Proposition 3 shows the importance of risk aversion for asset prices. Risk premia increase linearly with R^a near the steady state.[17] This link should not be too surprising: Propositions 1–2 describe the risk premium for the simplest gambles over household wealth, while Proposition 3 shows that the same coefficient applies to more general gambles over financial assets that may be correlated with aggregate variables such as w_t, r_t, and d_t.[18]

Proposition 3 also generalizes the intertemporal capital asset pricing model of Merton (1973) to the case of variable labor. In equation (42), the first term is R^a times the covariance of the asset price with household wealth, while the second term captures the asset's covariance with Merton's "changes in investment opportunities." The first term can vanish if households are Arrow-Pratt risk neutral (that is, risk neutral in a cross-sectional or capital asset pricing model sense), but the second term remains

[17] This relationship also holds for the more general case of Epstein-Zin preferences, where it is easier to imagine varying risk aversion while holding the covariances in the model constant. See Swanson (2009) and Rudebusch and Swanson (2009).

[18] Boldrin, Christiano, and Fisher (1997) argue that it is u_{11}/u_1 rather than V_{11}/V_1 that matters for the equity premium. As shown here and in the numerical example in Section IV, below, it is V_{11}/V_1—which includes the effects of the labor margin—that is crucial. Although Boldrin et al. hold R^a constant in their Section IV and their Figure 2, the intertemporal elasticity of substitution and hence risk-free rate volatility change greatly across the circles in that figure; thus, even though R^a is held constant in their Figure 2, the covariance terms in Proposition 3 change greatly, leading to variation in the equity premium.

nonzero because even a risk-neutral household can benefit from purchasing consumption and leisure when their prices are low—that is, when wages are low or interest rates are high. Thus, an asset that pays off well in those states of the world carries a higher price even for risk-neutral investors.

Finally, Proposition 3 implies that it is no harder or easier to match asset prices in a dynamic equilibrium model with labor than it is in such a model without labor. A given level of risk aversion in a DSGE model with labor, measured correctly, will generate just as large a risk premium as the same level of risk aversion in a DSGE model without labor, for a given set of model covariances. Thus, the equity premium is not any harder to match, or any more puzzling, in dynamic production models with endogenous labor supply than in models without it.

We conclude this section by noting that the risk premium is essentially linear in relative as well as absolute risk aversion, using an appropriate measure of covariance:

COROLLARY 4: *Given Assumptions* 1–7, *the risk premium in equation* (42) *can be written as*

$$R^c(a;\boldsymbol{\theta}) \cdot \mathrm{Cov}_t\left(dp_{t+1}, \frac{d\hat{A}_{t+1}}{A}\right) + \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}) \tag{43}$$

$$\text{or} \quad R^{cl}(a;\boldsymbol{\theta}) \cdot \mathrm{Cov}_t\left(dp_{t+1}, \frac{d\hat{A}_{t+1}}{\tilde{A}}\right) + \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}), \tag{44}$$

where A and $\tilde{A}$ are as in Definitions 2–3, *and $d\hat{A}_{t+1}$ and $d\Psi_{t+1}$ are as defined in Proposition* 7.

PROOF:

See Appendix.

IV. Numerical Examples

A. *Risk Aversion away from the Steady State*

The simple, closed-form expressions for risk aversion derived above hold exactly only at the model's nonstochastic steady state. For values of $(a_t;\boldsymbol{\theta}_t)$ away from steady state, these expressions are only approximations. In this section, we evaluate the accuracy of those approximations by computing risk aversion numerically for a standard real business cycle model.

There is a unit continuum of representative households in the model, each with optimization problem (1)–(3), with additively separable period utility function (34) from Example 2. The economy contains a unit continuum of perfectly competitive firms, each with production function $y_t = A_t k_t^{1-\alpha} l_t^{\alpha}$, where y_t, l_t, and k_t denote firm output, labor, and beginning-of-period capital, and A_t denotes an exogenous technology process that follows $\log A_t = \rho \log A_{t-1} + \varepsilon_t$, where ε_t is i.i.d. with mean zero and variance σ_ε^2. Labor and capital are supplied by households at the competitive wage and rental rates w_t and r_t^k. Capital is the only asset, which households accumulate according to $k_{t+1} = (1 + r_t)k_t + w_t l_t - c_t$, where $r_t = r_t^k - \delta$, δ is the capital depreciation rate, and c_t denotes household consumption.

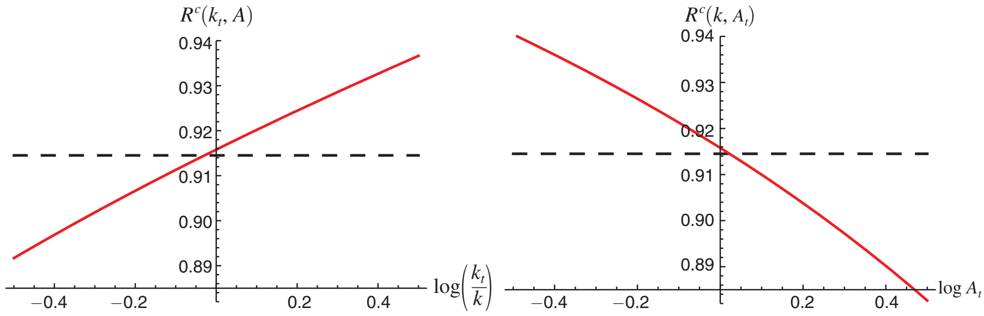

Figure 2

Notes: Coefficient of relative risk aversion $R^c(k_t,A_t)$ for period utility function $u(c_t,l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, with $\gamma = 2$ and $\chi = 1.5$, in a standard real business cycle model. Solid red lines depict numerical solution for $R^c(k_t,A_t)$, holding one state variable at a time fixed at its steady-state value. Dashed black lines depict the constant, closed-form solution for $R^c(k,A)$ for comparison. See text for details.

We set $\beta = 0.99$, $\gamma = 2$, and $\chi = 1.5$, corresponding to an intertemporal elasticity of substitution of 0.5 and Frisch elasticity of ⅔. We set $\eta = 0.4514$ to normalize steady-state labor $l = 1$. We set $\alpha = 0.7$, $\delta = 0.025$, $\rho = 0.9$, and $\sigma_\varepsilon = 0.01$.

The state variables of the model are k_t and A_t.[19] At the steady state, relative risk aversion is given by equation (35), which for the parameter values above implies $R^c(k,A) = 0.9145$, less than half the traditional measure of $\gamma = 2$. Away from steady state, equations (8) and (10)–(17) remain valid, and we use them to compute $R^c(k_t,A_t)$ by solving for V_1, V_{11}, λ_t, and $\partial c_t^*/\partial a_t$ numerically (see the Appendix for details). Figure 2 graphs the result over a wide range of values for k_t and A_t, ± 50 log percentage points (equal to about 15 and 20 standard deviations of $\log k_t$ and $\log A_t$, respectively). The solid red lines in the figure depict the solution for $R^c(k_t,A_t)$, while the horizontal dashed black lines depict the constant $R^c(k,A) = 0.9145$ for comparison. The key observation is that, even over the very wide range of values for (k_t,A_t) considered, the household's coefficient of relative risk aversion ranges between 0.88 and 0.94, very close to the steady-state value of 0.9145, and never near the traditional value of 2.[20] Thus, the closed-form expressions in Section II provide a good approximation to the true level of risk aversion in a standard model even far away from steady state.

B. *Risk Aversion and the Equity Premium*

The numerical exercise above uses parameter values that are typical of calibrations to macroeconomic data. It is well known, however, that this type of parameterization

[19] The endogenous state variable is k_t, while the exogenous state variables are A_t and K_t, the aggregate capital stock. In equilibrium, $k_t = K_t$, so we write the state vector as (k_t,A_t), although it would be written as $(k_t;A_t,K_t)$ for the analysis in Section II.

[20] The red lines do not intersect the black lines at the vertical axis because c_t^* and l_t^* evaluated at (k,A) do not equal the nonstochastic steady-state values c and l due to the presence of uncertainty (e.g., precautionary savings); one can add σ_ε to the exogenous state $\boldsymbol{\theta}_t$ to capture this difference formally. Also note that absolute (rather than relative) risk aversion is countercyclical with respect to both k_t and A_t, although this is not plotted due to space constraints. In Figure 2, relative risk aversion is procyclical with respect to k_t because household wealth increases with k_t, and for this example the increase in household wealth for higher k_t more than offsets the fall in absolute risk aversion.

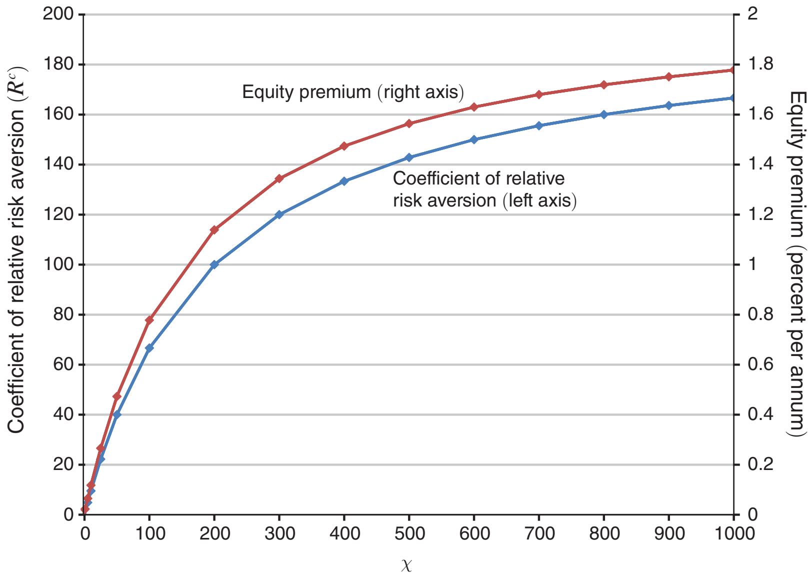

Figure 3

Notes: Coefficient of relative risk aversion $R^c(k,A)$ and the equity premium for period utility function $u(c_t,l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ with $\gamma = 200$, plotted as functions of χ, in a standard real business cycle model. The equity premium is proportional to risk aversion and both risk aversion and the equity premium approach 0 as χ approaches 0. See text for details.

produces a negligible equity premium (e.g., Mehra and Prescott 1985), amounting to less than one basis point for a claim to the aggregate consumption stream in the example above. In Figure 3, I consider a parameterization of the model in which the equity premium is larger, fixing $\gamma = 200$, and plot $R^c(k,A)$ and the equity premium on the left and right axes as functions of χ.[21] As predicted in Section III, the equity premium increases essentially linearly with risk aversion. Both $R^c(k,A)$ and the equity premium fall toward zero as χ approaches zero, despite the fact that $-cu_{11}/u_1 = \gamma$ is fixed at 200. These observations confirm that risk aversion as defined in the present paper—and not the traditional measure—is the proper concept for understanding asset prices in the model.

V. Balanced Growth

The results in the previous sections carry through essentially unchanged to the case of balanced growth. I collect the corresponding expressions here in Lemma 2, Proposition 4, and Corollary 5, and provide proofs in the Appendix.

Along a balanced growth path, $x \in \{l,r\}$ satisfies $x_{t+k} = x_t$ for $k = 1,2,\ldots$, and I drop the time subscript to denote the constant value. For $x \in \{a,c,w,d\}$, we have $x_{t+k} = G^k x_t$ for $k = 1,2,\ldots$, for some $G \in (0, 1+r)$, and I let x_t^{bg} denote the balanced growth path value. I denote the balanced growth path value of θ_t by θ_t^{bg}, although the

[21] For each value of χ, I set steady-state labor $l = 1$ by choosing η appropriately. See the Appendix for additional details of this computation.

elements of $\boldsymbol{\theta}$ may grow at different constant rates over time (or remain constant). Additional details regarding balanced growth are provided in King, Plosser, and Rebelo (1988, 2002).

LEMMA 2: *Given Assumptions* 1–6 *and* 7′, *for* $k = 1, 2, \ldots,$ *along the balanced growth path:* (*i*) $\lambda_{t+k}^{bg} = G^{-k}\lambda_t^{bg}$, *where* λ_t^{bg} *denotes the balanced growth path value of* λ_t; (*ii*) $\partial c_{t+k}^*/\partial a_t = G^k \partial c_t^*/\partial a_t$; (*iii*) $\partial l_{t+k}^*/\partial a_t = \partial l_t^*/\partial a_t$; *and* (*iv*) $\partial c_t^*/\partial a_t = (1 + r - G)/(1 + w_t^{bg}\lambda_t^{bg})$.

Note that $w_t^{bg}\lambda_t^{bg}$ in Lemma 2 is constant over time because w and λ grow at reciprocal rates. The larger is G, the smaller is $\partial c_t^*/\partial a_t$, since the household chooses to absorb a greater fraction of asset shocks in future periods.

PROPOSITION 4: *Given Assumptions* 1–6 *and* 7′, *absolute risk aversion satisfies*

$$R^a(a_t^{bg};\boldsymbol{\theta}_t^{bg}) = \frac{-V_{11}(a_{t+1}^{bg};\boldsymbol{\theta}_{t+1}^{bg})}{V_1(a_{t+1}^{bg};\boldsymbol{\theta}_{t+1}^{bg})} \tag{45}$$

and

$$R^a(a_t^{bg};\boldsymbol{\theta}_t^{bg}) = \frac{-u_{11} + \lambda_t^{bg}u_{12}}{u_1}\,\frac{\frac{1+r}{G} - 1}{1 + w_t^{bg}\lambda_t^{bg}}, \tag{46}$$

where u_{ij} *denotes the corresponding partial derivative of* u *evaluated at* (c_t^{bg}, l).

Note that equation (46) agrees with Proposition 2 when $G = 1$. The larger is G, the smaller is R^a, since larger G implies greater household wealth and ability to absorb asset shocks.

COROLLARY 5: *Given Assumptions* 1–6 *and* 7′, *relative risk aversion satisfies*

$$R^c(a_t^{bg};\boldsymbol{\theta}_t^{bg}) = \frac{-u_{11} + \lambda_t^{bg}u_{12}}{u_1}\,\frac{c_t^{bg}}{1 + w_t^{bg}\lambda_t^{bg}} \tag{47}$$

and

$$R^{cl}(a_t^{bg};\boldsymbol{\theta}_t^{bg}) = \frac{-u_{11} + \lambda_t^{bg}u_{12}}{u_1}\,\frac{c_t^{bg} + w_t^{bg}(\bar{l} - l)}{1 + w_t^{bg}\lambda_t^{bg}}. \tag{48}$$

Thus, the expressions for relative risk aversion are unchanged by balanced growth.

VI. Conclusions

The traditional measure of risk aversion, $-cu_{11}/u_1$, ignores the household's ability to partially offset shocks to income or asset values with changes in hours worked. For reasonable parameterizations, the traditional measure can overstate risk aversion by a factor of three or more. Many studies in the macroeconomics, macro-finance, and international literatures thus may overstate the actual degree of risk aversion in their models by a substantial degree. Studies using Hansen-type (1985) linear labor preferences for algebraic simplicity are also effectively assuming risk neutrality.[22]

Risk aversion matters for asset pricing. The equity premium and other risk premia are closely tied to risk aversion as defined in the present paper, and are essentially unrelated to $-cu_{11}/u_1$. Risk aversion and risk premia in these models can be essentially zero even when the traditional measure of risk aversion is large.

Risk aversion and the intertemporal elasticity of substitution are nonreciprocal. This observation may be useful for model calibration since, e.g., high values of γ in $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ are not ruled out by empirical estimates of risk aversion.

The insights of the present paper are general and apply to Epstein-Zin (1989) preferences and internal and external habits as well as time-separable expected utility. Swanson (2009) provides extensions of the results and formulae in the present paper to those cases.

It is also worth noting two nonimplications of the present paper. First, I do not find that it is any harder or easier to match risk premia in dynamic equilibrium models with labor than in models without labor (Proposition 3). Second, I do not shed any light on what plausible empirical values for risk aversion might be. Empirical estimates of risk aversion based on surveys, changes in income or wealth, or cash prizes are generally just as valid in the present framework as they are in dynamic models without labor.

Finally, many of the observations of the present paper apply not just to dynamic models with labor, but to any such model with multiple goods in the utility function. Models with home production, money in the utility function, or tradable and nontradable goods can all imply very different household attitudes toward risk than traditional measures of risk aversion would suggest. The simple, closed-form expressions for risk aversion derived in this paper, and the methods of the paper more generally, are potentially useful in any of these cases, in pricing any asset—stocks, bonds, or futures, in foreign or domestic currency—within the framework of dynamic equilibrium models. Since these models are a mainstay of research in academia, at central banks, and international financial institutions, the applicability of the results should be widespread.

Appendix: Proofs of Propositions and Numerical Solution Details

PROOF OF PROPOSITION 1:

Since $(a_t; \boldsymbol{\theta}_t)$ is an interior point of X, $V(a_t + \frac{\sigma \underline{\varepsilon}}{1+r_t}; \boldsymbol{\theta}_t)$ and $V(a_t + \frac{\sigma \bar{\varepsilon}}{1+r_t}; \boldsymbol{\theta}_t)$ exist for sufficiently small σ, and $V(a_t + \frac{\sigma \underline{\varepsilon}}{1+r_t}; \boldsymbol{\theta}_t) \leq \tilde{V}(a_t; \boldsymbol{\theta}_t; \sigma) \leq V(a_t + \frac{\sigma \bar{\varepsilon}}{1+r_t}; \boldsymbol{\theta}_t)$, hence $\tilde{V}(a_t; \boldsymbol{\theta}_t; \sigma)$ exists. Moreover, since $V(\cdot; \cdot)$ is continuous and increasing in its first argument, the intermediate value theorem implies that there exists a unique $-\mu(a_t; \boldsymbol{\theta}_t; \sigma) \in [\sigma \underline{\varepsilon}, \sigma \bar{\varepsilon}]$ satisfying $V(a_t - \frac{\mu}{1+r_t}; \boldsymbol{\theta}_t) = \tilde{V}(a_t; \boldsymbol{\theta}_t; \sigma)$.

[22] Examples include Lagos and Wright (2005); Khan and Thomas (2008); Bachmann, Caballero, and Engel (2010); and Bachmann and Bayer (2009).

For a sufficiently small fee μ in equation (7), the change in household welfare (5) is given to first order by

$$\text{(A1)} \qquad \frac{-V_1(a_t;\boldsymbol{\theta}_t)}{1+r_t}\,d\mu.$$

Using the envelope theorem, we can rewrite (A1) as

$$\text{(A2)} \qquad -\beta E_t V_1(a^*_{t+1};\boldsymbol{\theta}_{t+1})\,d\mu.$$

Turning now to the gamble in equation (6), note that the household's optimal choices for consumption and labor in period t, c^*_t and l^*_t, will generally depend on the size of the gamble σ—for example, the household may undertake precautionary saving when faced with this gamble. Thus, in this section we write $c^*_t \equiv c^*(a_t;\boldsymbol{\theta}_t;\sigma)$ and $l^*_t \equiv l^*(a_t;\boldsymbol{\theta}_t;\sigma)$ to emphasize this dependence on σ. The household's value function, inclusive of the one-shot gamble in equation (6), satisfies

$$\text{(A3)} \qquad \tilde{V}(a_t;\boldsymbol{\theta}_t;\sigma) \;=\; u(c^*_t,l^*_t) \;+\; \beta E_t V(a^*_{t+1};\boldsymbol{\theta}_{t+1}),$$

where $a^*_{t+1} \equiv (1+r_t)a_t + w_t l^*_t + d_t - c^*_t$. Because equation (6) describes a one-shot gamble in period t, it affects assets a^*_{t+1} in period $t+1$ but otherwise does not affect the household's optimization problem from period $t+1$ onward; as a result, the household's value-to-go at time $t+1$ is just $V(a^*_{t+1};\boldsymbol{\theta}_{t+1})$, which does not depend on σ except through a^*_{t+1}.

Differentiating equation (A3) with respect to σ, the first-order effect of the gamble on household welfare is

$$\text{(A4)} \qquad \left[u_1\frac{\partial c^*}{\partial\sigma} \;+\; u_2\frac{\partial l^*}{\partial\sigma} \;+\; \beta E_t V_1 \cdot \left(w_t\frac{\partial l^*}{\partial\sigma} \;-\; \frac{\partial c^*}{\partial\sigma} \;+\; \varepsilon_{t+1}\right)\right] d\sigma,$$

where the arguments of u_1, u_2, and V_1 are suppressed to reduce notation. Optimality of c^*_t and l^*_t implies that the terms involving $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ in (A1) cancel, as in the usual envelope theorem (these derivatives vanish at $\sigma = 0$ anyway, for the reasons discussed below). Moreover, $E_t V_1(a^*_{t+1};\boldsymbol{\theta}_{t+1})\varepsilon_{t+1} = 0$ because ε_{t+1} is independent of $\boldsymbol{\theta}_{t+1}$ and a^*_{t+1}, evaluating the latter at $\sigma = 0$. Thus, the first-order cost of the gamble is zero, as in Arrow (1971) and Pratt (1964).

To second order, the effect of the gamble on household welfare is

$$\text{(A5)} \qquad \left[u_{11}\left(\frac{\partial c^*}{\partial\sigma}\right)^2 \;+\; 2u_{12}\frac{\partial c^*}{\partial\sigma}\frac{\partial l^*}{\partial\sigma} \;+\; u_{22}\left(\frac{\partial l^*}{\partial\sigma}\right)^2 \;+\; u_1\frac{\partial^2 c^*}{\partial\sigma^2} \;+\; u_2\frac{\partial^2 l^*}{\partial\sigma^2}\right.$$
$$+\; \beta E_t V_{11} \cdot \left(w_t\frac{\partial l^*}{\partial\sigma} \;-\; \frac{\partial c^*}{\partial\sigma} \;+\; \varepsilon_{t+1}\right)^2$$
$$\left. +\; \beta E_t V_1 \cdot \left(w_t\frac{\partial^2 l^*}{\partial\sigma^2} \;-\; \frac{\partial^2 c^*}{\partial\sigma^2}\right)\right]\frac{d\sigma^2}{2}.$$

The terms involving $\partial^2 c^*/\partial\sigma^2$ and $\partial^2 l^*/\partial\sigma^2$ cancel due to the optimality of c_t^* and l_t^*. The derivatives $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ vanish at $\sigma = 0$. (There are two ways to see this: first, the linearized version of the model is certainty equivalent; alternatively, for symmetric E, the gamble in equation (6) is isomorphic for positive and negative σ, hence c^* and l^* must be symmetric about $\sigma = 0$, implying the derivatives vanish.) Thus, for infinitesimal gambles, equation (A5) simplifies to

$$\beta E_t V_{11}(a_{t+1}^*; \boldsymbol{\theta}_{t+1})\, \varepsilon_{t+1}^2 \frac{d\sigma^2}{2}. \tag{A6}$$

Finally, ε_{t+1} is independent of $\boldsymbol{\theta}_{t+1}$ and a_{t+1}^*, evaluating the latter at $\sigma = 0$. Since ε_{t+1} has unit variance, equation (A6) reduces to

$$\beta E_t V_{11}(a_{t+1}^*; \boldsymbol{\theta}_{t+1}) \frac{d\sigma^2}{2}. \tag{A7}$$

Equating (A2) to (A7) allows us to solve for $d\mu$ as a function of $d\sigma^2$. Thus, $\lim_{\sigma\to 0} 2\mu(a_t; \boldsymbol{\theta}_t; \sigma)/\sigma^2$ exists and is given by

$$\frac{-E_t V_{11}(a_{t+1}^*; \boldsymbol{\theta}_{t+1})}{E_t V_1(a_{t+1}^*; \boldsymbol{\theta}_{t+1})}. \tag{A8}$$

To evaluate (A8) at the nonstochastic steady state, set $a_{t+1} = a$ and $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}$ to get

$$\frac{-V_{11}(a; \boldsymbol{\theta})}{V_1(a; \boldsymbol{\theta})}. \tag{A9}$$

PROOF OF LEMMA 1:

Differentiating the household's Euler equation (15) and evaluating at steady state yields

$$u_{11}(dc_t^* - E_t dc_{t+1}^*) + u_{12}(dl_t^* - E_t dl_{t+1}^*) = \beta E_t u_1 dr_{t+1}, \tag{A10}$$

which, applying equation (39), becomes

$$(u_{11} - \lambda u_{12})(dc_t^* - E_t dc_{t+1}^*) - \frac{u_1 u_{12}}{u_{22} + w u_{12}}(dw_t - E_t dw_{t+1}) = \beta E_t u_1 dr_{t+1}. \tag{A11}$$

Note that equation (A11) implies, for each $k = 1, 2, \ldots,$

$$E_t dc_{t+k}^* = dc_t^* - \frac{u_1 u_{12}}{u_{11} u_{22} - u_{12}^2}(dw_t - E_t dw_{t+k}) - \frac{\beta u_1}{u_{11} - \lambda u_{12}} E_t \sum_{i=1}^{k} dr_{t+i}. \tag{A12}$$

Combining equations (2)–(3), differentiating, and evaluating at steady state yields

$$\text{(A13)} \qquad E_t \sum_{k=0}^{\infty} \frac{1}{(1 + r)^k} (dc^*_{t+k} - w dl^*_{t+k} - l dw_{t+k} - dd_{t+k} - a dr_{t+k}) = (1 + r) da_t.$$

Substituting equations (39) and (A12) into (A13), and solving for dc^*_t, yields

$$\text{(A14)} \quad dc^*_t = \frac{r}{1+r} \frac{1}{1+w\lambda} \left[(1+r) da_t + E_t \sum_{k=0}^{\infty} \frac{1}{(1 + r)^k} (l dw_{t+k} + dd_{t+k} + a dr_{t+k}) \right] + \frac{u_1 u_{12}}{u_{11} u_{22} - u_{12}^2} dw_t + \frac{1}{1+r} \frac{-u_1}{u_{11} - \lambda u_{12}} E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} \left[\frac{r\lambda}{1 + w\lambda} dw_{t+k} - \beta dr_{t+k+1} \right].$$

PROOF OF COROLLARY 4:

From Definition 3, $\tilde{A}_t \equiv (1 + r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} (c^*_\tau + w_\tau(\bar{l} - l^*_\tau))$. Evaluated at steady state, $r\tilde{A} = c + w(\bar{l} - l)$, hence equation (44) follows from (42). In the same way, Definition 2 and equation (42) imply (43).

Numerical Solution Procedure for Section IV: The equations of the model itself are standard:

$$\text{(A17)} \qquad Y_t = A_t K_{t-1}^{1-\alpha} L_t^{\alpha},$$

$$\text{(A18)} \qquad K_t = (1 - \delta) K_{t-1} + Y_t - C_t,$$

$$\text{(A19)} \qquad C_t^{-\gamma} = \beta E_t (1 + r_{t+1}) C_{t+1}^{-\gamma},$$

$$\text{(A20)} \qquad \eta L_t^{\chi} / C_t^{-\gamma} = w_t,$$

$$\text{(A21)} \qquad r_t = (1 - \alpha) Y_t / K_{t-1} - \delta,$$

$$\text{(A22)} \qquad w_t = \alpha Y_t / L_t,$$

$$\text{(A23)} \qquad \log A_t = \rho \log A_{t-1} + \varepsilon_t.$$

In equations (A17)–(A23), note that K_{t-1} denotes the capital stock at the beginning of period t (or the end of period $t - 1$), so the notation differs slightly from the main text for compatibility with the numerical algorithm below. To compute risk aversion, I need to append the following auxiliary variables and equations to (A17)–(A23):

$$\lambda_t = (\gamma/\chi)L_t/C_t, \tag{A24}$$

$$C_t^{-\gamma-1}DCDA_t = \beta E_t(1 + r_{t+1})C_{t+1}^{-\gamma-1} \times DCDA_{t+1}[(1 + r_t) - (1 + w_t\lambda_t)DCDA_t], \tag{A25}$$

$$CARA_t = E_t(1 + r_{t+1})\gamma C_{t+1}^{-\gamma-1}DCDA_{t+1}/(C_t^{-\gamma}/\beta), \tag{A26}$$

$$PDVC_t = C_t + \beta E_t(C_{t+1}^{-\gamma}/C_t^{-\gamma})PDVC_{t+1}, \tag{A27}$$

$$CRRA_t = CARA_t PDVC_t/(1 + r_t). \tag{A28}$$

Equation (A24) corresponds to equation (14), (A25) to (17), (A26) to Proposition 1, and (A27)–(A28) to Definition 2. The variable $DCDA_t$ corresponds to $\partial c_t^*/\partial a_t$. Note that

$$\frac{\partial c_{t+1}^*}{\partial a_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}^*}\left[(1+r_t) - w_t\lambda_t\frac{\partial c_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t}\right], \tag{A29}$$

which I use in equation (A25). I use the envelope condition $V_1(a_t;\boldsymbol{\theta}_t) = \beta(1+r_t) \times E_t V_1(a_{t+1};\boldsymbol{\theta}_{t+1})$ to rewrite $E_t V_1(a_{t+1};\boldsymbol{\theta}_{t+1})$ in (A26), and equations (10)–(11) to rewrite V_1 and V_{11} in terms of derivatives of u.

I solve equations (A17)–(A28) numerically using the Perturbation AIM algorithm of Swanson, Anderson, and Levin (2006) to compute second- through seventh-order Taylor series approximate solutions to equations (A17)–(A28) around the nonstochastic steady state. These are guaranteed to be arbitrarily accurate in a neighborhood of the nonstochastic steady state, but importantly also converge globally within the domain of convergence of the Taylor series as the order of the approximation becomes large. Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) solve a standard real business cycle model like equations (A17)–(A23) using a variety of numerical methods, including second- and fifth-order perturbation, and find that the perturbation solutions are among the most accurate methods globally, as well as being the fastest to compute. The perturbation solutions I compute for equations (A17)–(A28) are indistinguishable from one another after the third order over the range of values considered in Figure 2, consistent with Taylor series convergence, so I report only the seventh-order solution in Figure 2.

The equity premium in the model is computed as

$$p_t = \beta E_t(C_{t+1}^{-\gamma}/C_t^{-\gamma})(C_{t+1} + p_{t+1}), \tag{A30}$$

(A31) $$1/(1 + r_t^f) = \beta E_t(C_{t+1}^{-\gamma}/C_t^{-\gamma}),$$

(A32) $$ep_t = E_t(C_{t+1} + p_{t+1})/p_t - (1 + r_t^f),$$

where p_t denotes the price of equity, r_t^f the risk-free rate, and ep_t the equity premium. These equations are combined with (A17)–(A28), solved to seventh order, and evaluated at the nonstochastic steady state to produce the results in Figure 3.

PROOF OF LEMMA 2:

(i) The household's Euler equation implies

(A33) $$u_1(c_t^{bg}, l) = \beta(1 + r)u_1(c_{t+1}^{bg}, l) = \beta(1 + r)u_1(Gc_t^{bg}, l).$$

Similarly, for labor,

(A34) $$u_2(c_t^{bg}, l) = \beta(1 + r)\frac{w_t^{bg}}{w_{t+1}^{bg}}u_2(c_{t+1}^{bg}, l) = \beta(1 + r)G^{-1}u_2(Gc_t^{bg}, l).$$

As in King, Plosser, and Rebelo (2002), I assume that preferences u are consistent with balanced growth for all initial asset stocks and wages in a neighborhood of a_t^{bg} and w_t^{bg}, and hence for all initial values of (c_t, l_t) in a neighborhood of (c_t^{bg}, l). Thus, we can differentiate equations (A33) and (A34) to yield

(A35) $$u_{11}(c_t^{bg}, l) = \beta(1 + r)G u_{11}(Gc_t^{bg}, l),$$

(A36) $$u_{12}(c_t^{bg}, l) = \beta(1 + r)u_{12}(Gc_t^{bg}, l),$$

(A37) $$u_{22}(c_t^{bg}, l) = \beta(1 + r)G^{-1}u_{22}(Gc_t^{bg}, l).$$

Applying equations (A35)–(A37) to equation (14),

(A38) $$\lambda_{t+1}^{bg} = \frac{w_{t+1}^{bg}u_{11}(c_{t+1}^{bg}, l) + u_{12}(c_{t+1}^{bg}, l)}{u_{22}(c_{t+1}^{bg}, l) + w_{t+1}^{bg}u_{12}(c_{t+1}^{bg}, l)} = G^{-1}\lambda_t^{bg}.$$

(ii) Assumptions 1–5 imply equations (10)–(17) in the text. Hence:

(A39) $$\left(u_{11}(c_t^{bg}, l) - \lambda_t^{bg}u_{12}(c_t^{bg}, l)\right)\frac{\partial c_t^*}{\partial a_t}$$
$$= \beta(1 + r)\left(u_{11}(c_{t+1}^{bg}, l) - \lambda_{t+1}^{bg}u_{12}(c_{t+1}^{bg}, l)\right)\frac{\partial c_{t+1}^*}{\partial a_t}.$$

Solving for $\partial c^*_{t+1}/\partial a_t$ and using equations (A35)–(A38) yields $\partial c^*_{t+1}/\partial a_t = G\partial c^*_t/\partial a_t$.

(iii) Follows from equations (13), (A35)–(A38), and (ii).

(iv) Use the household's budget constraint (2)–(3) and (ii) to solve for $\partial c^*_t/\partial a_t$.

PROOF OF PROPOSITION 4:

Proposition 1 implies equation (45). Assumptions 1–5 imply equations (10)–(17). Substituting equations (10), (11), (13)–(14), and Lemma 9(iv) into (45), we have

$$\text{(A40)} \quad R^a(a_t^{bg};\theta_t^{bg}) = \frac{-u_{11}(c_{t+1}^{bg},l) + \lambda_{t+1}^{bg}\,u_{12}(c_{t+1}^{bg},l)}{u_1(c_{t+1}^{bg},l)}\,\frac{1+r-G}{1+w_{t+1}^{bg}\,\lambda_{t+1}^{bg}}.$$

Using equations (A35)–(A38) and Lemma 2 completes the proof.

PROOF OF COROLLARY 5:

As in Definitions 2–3, I define wealth A_t^{bg} in beginning- rather than end-of-period-t units; this requires multiplying by $(1+r)^{-1}G^{-1}$ rather than just $(1+r)^{-1}$. Then the present discounted value of consumption along the balanced growth path is given by $A_t^{bg} = c_t^{bg}/\left(\frac{1+r}{G} - 1\right)$, and the present discounted value of leisure by $w_t^{bg}(\bar{l} - l)/\left(\frac{1+r}{G} - 1\right)$.

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