Optimal Time-Consistent Monetary Policy in the New Keynesian Model with Repeated Simultaneous Play

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Abstract

We solve for the optimal time-consistent monetary policy in the New Keynesian model with repeated simultaneous play between the monetary authority, households, and firms. Recent work on optimal time-consistent monetary policy has emphasized the existence of multiple Markov perfect equilibria in the New Keynesian model (e.g., King and Wolman, 2004). In this paper, we show that this multiplicity is not intrinsic to the New Keynesian model itself, but is instead driven by a special timing assumption by previous authors that play is "repeated Stackelberg"—in which the monetary authority must pre-commit each period to a value for the monetary instrument—as opposed to repeated simultaneous, in which the monetary authority and the private sector determine the economic equilibrium simultaneously and jointly every period. To illlustrate this, we derive a closed-form solution for the set of all possible Markov perfect equilibria in the two-period Taylor contracting version of the New Keynesian model under repeated simultaneous play and show that the equilibrium in that model is unique.

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1 Introduction

Many countries have witnessed periods of high and persistent inflation, as the U.S. did in the 1970s. What caused these episodes? More importantly, how can central banks or governments prevent them from recurring in the future?

One prominent explanation, due originally to Barro and Gordon (1983) and refined by Chari, Christiano, and Eichenbaum (1998), is that the time-consistency problem of monetary policy in the absence of a commitment mechanism (Kydland and Prescott, 1977) leads to a large multiplicity of possible "discretionary" equilibria, some with substantially higher inflation rates than others. Two criticisms have been raised against this hypothesis in the literature, however. First, the hypothesis has little empirical content, because the enormous number and range of equilibria it allows make it essentially impossible to reject or critique the hypothesis on the basis of observation. Second, many of the equilibria are complex and require a great deal of sophistication and coordination across a large number of atomistic agents in order to arise.

In response to these criticisms, the literature has turned to the much simpler and smaller class of *Markov perfect equilibria*, in which agents may only condition their actions on economic fundamentals—i.e., the state variables of the model. A striking finding of this literature (e.g., Albanesi, Chari, and Christiano, 2003, King and Wolman, 2004, Siu, 2006, and Armenter, 2006) is that there exist multiple *Markov perfect* equilibria in standard, New Keynesian dynamic general equilibrium models. As a result, these studies conclude that the U.S. and other countries could once again find themselves caught in a bad "expectations trap" for inflation and a possible repeat of the 1970s experience. Moreover, these studies raise the possibility that the linear-quadratic approximation to the New Keynesian model that has been widely used to compute the optimal time-consistent monetary policy (e.g., Clarida, Gali, and Gertler, 1999, Svensson and Woodford, 2003, Woodford, 2003) is completely missing one of the most important features of the New Keynesian model under discretion.

In this paper, we show that multiple Markov perfect equilibria are not intrinsic to the New Keynesian model itself, but instead can be due to an auxiliary assumption about the timing of play by the agents in the model which is sometimes not made explicit. The main contribution of this paper is to clarify the importance of within-period timing of play in New Keynesian and other monetary dynamic general equilibrium models. We analyze a specific prominent example—the New Keynesian model of King and Wolman (2004)—and show that the equilibrium is unique in their model under "repeated simultaneous" play, while there are multiple Markov perfect equilibria under a "repeated Stackelberg" timing assumption, which was assumed by King and Wolman. This result suggests that the timing assumptions should be stated explicitly as part of the environment and justified on economic grounds, as it can be in at least some cases the only source of multiplicity in the model.

The literature on monetary Markov perfect equilibria has generally made use of two notions of time consistency involving two different within-period timing of actions. First, there is a large body of work studying optimal time-consistent policy in linear-quadratic models, in which the within-period timing of play is generally simultaneous between the government and the private sector. This assumption is made most explicitly in papers which allow for stochastic shocks and imperfect observation by policymakers and the private sector of the true state of the economy, such as in the papers by Svensson and Woodford (2003), Woodford (2003), and Pearlman (1992). In this literature, there is typically a unique Markov perfect equilibrium. The second branch of the monetary literature emphasizes multiple equilibria but uses a different within-period timing assumption. In particular, Albanesi, Chari, and Christiano (2003), Khan, King, and Wolman (2001), Dedola (2002), King and Wolman (2004), Siu (2005), and Armenter (2005) all use a timing assumption in which each period is divided into two halves, with the monetary authority "precommitting" to a value for the policy instrument (an interest rate or the money supply) in the first half of the period, and the private sector responding in the second half of the period in a repeated Stackelberg fashion. Under this within-period timing assumption, multiple equilibria arise in a broad class of models. A common claim is that this multiplicity is intrinsic to the New Keynesian model and that the earlier literature failed to find this due to the linear-quadratic approximation methodology. Our analysis suggests that instead the key difference is the timing of play within each period.

Ortigueira (2005) and Cohen and Michel (1988) observe that the literature on optimal fiscal policy has also proceeded under two different within-period timing assumptions. On the one hand, Turnovsky and Brock (1980) and Judd (1988, Ch 16) assume simultaneous play by the private sector and the government in every period, while Klein, Krusell, and Rios-Rull (2004) assume that the government plays first in each period in a Stackelberg fashion. (Note that the definition of "simultaneous play" in these models is simpler than for our New Keynesian general equilibrium model below; in particular, it is possible in the simple fiscal model to define some of the endogenous variables as "residuals", which ensures that the model's aggregate resource constraints are always respected automatically. In our New Keynesian model below, this issue is more complicated.) Ortigueira shows that these differences in within-period timing lead to different equilibrium values of the endogenous variables and welfare. Ortigueira, however, does not discuss the implication of within-period timing on the multiplicity of equilibria. Our finding suggests that within-period precommitment increases the dynamic inconsistency problem by introducing multiplicity. The key reason in our setting is that if the government moves first, this curbs its ability to react to the behavior of the private sector in that period, limiting its ability to exclude self-fulfilling prophecies.

We argue that the assumption of repeated simultaneous play by the monetary authority is more appealing than that of repeated Stackelberg play for at least two economic reasons. First, when the central bank and the private sector play simultaneously, it no longer matters whether the central bank's instrument is the money stock or the nominal interest rate—the set of equilibria are identical under either assumption—

an equivalence which does not hold under repeated Stackelberg play, as shown in Dotsey and Hornstein (2006) and as we discuss in section 5, below. We regard this equivalence between monetary instruments as an appealing feature of our timing assumption because most central banks in practice adjust the money supply to maintain a short-term interest rate target, and it is not at all clear which should be regarded as the monetary policy instrument if the two are not equivalent. Second, central banks around the world continuously monitor and maintain a target for a short-term interest rate (or monetary aggregate) and are free to continuously adjust this target should unforeseen economic developments arise over the course of the month or quarter; thus, the idea of central banks "pre-committing" to a given level of the money supply or an interest rate is arguably at odds with the data.[1] Although one can address this second criticism by shrinking the length of a period in the Stackelberg model down to one day or even one hour, it is still not clear why one would want to treat the central bank and the private sector so asymmetrically as to have one or the other always play first, as opposed to having them determine the economic equilibrium simultaneously and jointly. Moreover, shrinking the length of a period does not address the first criticism above.

The remainder of the paper proceeds as follows. Section 2 defines the New Keynesian model of the private sector economy and the necessary conditions for a private sector Markov perfect equilibrium in that economy given an exogenous Markov interest rate process. Section 3 presents the optimal policy problem for the central bank and derives the necessary conditions that a Markov perfect equilibrium between the central bank, households, and firms must satisfy. Section 4 derives the closed-form solution to the model and proves the uniqueness of the equilibrium. Section 5 briefly introduces money into the model and shows that our results are all unchanged if the monetary policy instrument is the money supply instead of the short-term nominal interest rate. Section 6 discusses the differences between repeated simultaneous and repeated Stackelberg timing assumptions. Section 7 concludes. An Appendix contains detailed proofs of the propositions in Section 4.

2 The Private Sector Economy and Private Sector Equilibrium

We focus in this paper on a standard dynamic New Keynesian model of the economy with monopolistically competitive intermediate goods markets, two-period Taylor price contracts, perfectly competitive factor markets, flexible wages, and homogeneous labor. For tractability and simplicity, we abstract away from endogenous variation in the capital stock.

Before turning to the question of optimal monetary policy, it will be useful to first define the New Keynesian economy and the corresponding game Γ_0 for the case of an *exogenous i.i.d.* interest rate process

[1] Although we rarely observe central banks changing their instrument between regularly scheduled meetings along the equilibrium path, this should not be interpreted as a structural constraint on their feasible set of policies or their out-of-equilibrium behavior.

$\{r_t\}$.[2] This will allow us to address a number of key issues without the additional complication of having the central bank as an additional player in the game. We will defer until the next section the game Γ_1 that we are primarily interested in, in which the short-term nominal interest rate is set by an optimizing central bank.

Time in the game Γ_0 is discrete and continues forever. Play of the game Γ_0 begins in an initial period t_0, but we assume that there is an infinite history of variables, extending back before t_0, on which players may condition their actions (although this will turn out not to be a Markov perfect equilibrium outcome). We now define the players, payoffs, information sets, and action spaces of the game Γ_0.

2.1 Players in the Game Γ_0

2.1.1 Firms

There is a continuum of atomistic firms in the economy indexed by $i \in [0, 1)$. The measure of firms is constant over time. Each firm is a player in the game Γ_0. At each time t, each firm produces a single, differentiated product, also indexed by i, according to the linear production function:

$$y_t(i) = l_t(i),$$

where $y_t(i)$ is the quantity of output produced and $l_t(i)$ the quantity of labor employed by firm i in period t.[3]

The price of each good i must be set for two periods in a staggered Taylor fashion, with firms in $[0, 1/2)$ free to change their price in even periods and firms in $[1/2, 1)$ free to change their price in odd periods. Each firm i must satisfy demand for its product in every period at its posted price $p_t(i)$, hiring whatever labor inputs are necessary. Nominal profits for firm i in period t are given by:

$$\Pi_t(i) \equiv p_t(i)y_t(i) - w_t y_t(i), \tag{1}$$

where w_t is the nominal wage in period t. Firms are owned by households, below, and distribute all profits or losses to all households equally every period. We define aggregate firm profits by:

$$\Pi_t \equiv \int_0^1 \Pi_t(i)di. \tag{2}$$

2.1.2 Households

The economy also contains a continuum of atomistic households indexed by $j \in [0, 1]$. Each household is a player in the game Γ_0. There is no population growth. At each date t, each household j receives a utility

[2] Although the i.i.d. assumption may seem restrictive here, it will turn out to be general enough to extend naturally to the game Γ_1, below, in which we consider an optimizing central bank that chooses a value for r_t each period.

[3] There is no loss of generality in assuming linearity (as opposed to homotheticity of lower degree) because households' disutility of working, defined below, will be homothetic with arbitrary parameter χ.

flow according to:

$$\frac{C_t(j)^{1-\varphi} - 1}{1-\varphi} - \chi_0 \frac{L_t(j)^{1+\chi}}{1+\chi}, \tag{3}$$

where $C_t(j)$ is the quantity of the final good consumed and $L_t(j)$ the quantity of labor supplied by household j in period t. The household discounts future utility flows at the rate β per period. Households can buy and sell risk-free one-period nominal discount bonds which pay one dollar at maturity and have price $1/(1+r_t)$ in period t, where r_t is the one-period nominal interest rate. The household faces an intertemporal budget constraint defined by the asset accumulation equation:

$$B_t(j)/(1+r_t) = B_{t-1}(j) + w_t L_t(j) + \Pi_t - P_t C_t(j), \tag{4}$$

where Π_t is the household's aliquot share of aggregate firm profits, w_t denotes the nominal wage, P_t the price of the consumption good, and $B_t(j)$ the household's stock of one-period nominal bonds at the end of period t. To exclude Ponzi schemes we assume that the household can never borrow more than $\bar{b}$ so that

$$\frac{B_t(j)}{P_t} \leq \bar{b} \tag{5}$$

In the equilibria we consider, this bound will never be binding. We also define aggregate labor supply by:

$$L_t \equiv \int_0^1 L_t(j)dj. \tag{6}$$

2.2 Two Mechanisms (not Players) in the Game Γ_0

The economy also consists of two mechanisms—which are *not* players in the game Γ_0—that are not formally necessary but help to clarify the exposition of the model and the game. For example, one can define a continuum of competitive final good-producing firms as players in the game Γ_0, but then those additional players (which are uninteresting) must be carried all the way through the formal definition of the game.[4] Rather than distract attention from the essential features of the game, we simply assume that the aggregation of intermediate goods into final goods happens automatically and non-strategically, as specified below.

2.2.1 Competitive Goods Aggregator

The economy has a competitive *final good aggregator*—that is *not* a player in the game Γ_0—that demands intermediate goods i and transforms them into the final consumption good according to:

$$Y_t \equiv \left[\int_0^1 y_t(i)^{1/(1+\theta)} di \right]^{1+\theta}, \tag{7}$$

[4] Alternatively, one can drop the final good entirely and define the household's utility function directly in terms of the individual goods i rather than an aggregate consumption good, but then the action space of each household must be defined as a function space over all the individual goods, which introduces additional and uninteresting complications to the problem.

where Y_t denotes the quantity of final good produced from the intermediate goods $y_t(i)$. From the point of view of the game Γ_0, this transformation happens automatically. The transformation is perfectly competitive in the sense that the intermediate good demands satisfy:

$$y_t(i) = \left(\frac{p_t(i)}{P_t}\right)^{-(1+\theta)/\theta} Y_t, \tag{8}$$

where P_t is the Dixit-Stiglitz aggregate price index,

$$P_t \equiv \left[\int_0^1 p_t(i)^{-1/\theta} di\right]^{-\theta}. \tag{9}$$

2.2.2 Walrasian Auctioneer

The economy also has a Walrasian auctioner—that is *not* a player in the game Γ_0—that ensures that all markets clear at each time t. Although the introduction of an auctioneer may seem like a distraction, it makes explicit the mechanism by which the economy's aggregate resource constraints are enforced, as we will discuss in a moment, and makes simultanous play by a continuum of actors *possible*.

It is worth reminding the reader that a Walrasian auctioneer, or some alternative coordination mechanism, is always lurking in the background in standard dynamic general equilibrium models in macroeconomics. It is this auctioneer that allows firms and households to condition their play at time t on aggregate variables also dated t, such as the wage w_t, the aggregate price level P_t, and the interest rate r_t, without violating aggregate resource constraints. We introduce this auctioneer explicitly because once we introduce the monetary authority in the next section, we will also allow it to condition its action r_t on all the aggregate variables dated t. In isolation this assumption might strike one as odd, since it is contrary to some of the earlier literature, but once we recall that exactly the same assumption applies to households and firms—and is in fact necessary to derive the equilibrium—the assumption is quite natural. It seems instead somewhat unappealing to assume that while households and firms can condition their actions on aggregate variables dated at time t, the monetary authority cannot do so.

Firms and households submit price *schedules* and consumption demand and labor supply *schedules* to the auctioneer, where these schedules are functions of relevant variables dated t, such as the wage w_t, aggregate price level P_t, interest rate r_t, aggregate output Y_t, and so on (this is specified in more detail below). Given these schedules, the auctioneer then finds the equilibrium values of w_t, $p_t(i)$, $y_t(i)$, $l_t(i)$, $C_t(j)$, $L_t(j)$, and $B_t(j)$ for all i and j that clear the final good market, the labor market, and the bond market at time t, i.e.:

$$\int_0^1 C_t(j)dj = Y_t, \tag{10}$$

$$\int_0^1 l_t(i)di = L_t, \tag{11}$$

and

$$\int_0^1 B_t(j)dj = B_t = 0. \tag{12}$$

The auctioneer also determines the stochastic pricing kernel $Q_{t,t+1}$ which prices all state-contingent claims at date $t+1$ in monetary units at date t.[5] If the auctioneer cannot find an equilibrium that satisfies (10)–(12) in period t given the schedules submitted by all the players of the game, then the auctioneer sets $y_t(i)$, $l_t(i)$, $C_t(j)$, $L_t(j)$ to zero (this outcome is sufficiently costly to all the players that it is never an equilibrium of the game Γ_0).

It is interesting to contrast the game Γ_0 here (and the game Γ_1 below) with standard games of industry Bertrand or Cournot competition, in which firms may set whatever prices or produce whatever quantities they desire, subject only to their own technological constraints. Here, the game comprises the *entire economy*, and if a positive measure of firms or workers were to deviate from equilibrium play—or if a large player such as the central bank were to deviate—then the economy's aggregate resource constraints (10)–(12) could be violated without the intervention of the Walrasian auctioneer. This is easiest to see for the case of a large player such as the central bank in the game Γ_1: imagine introducing money into the model above, with interest rates at time t determined by household money demand and by the quantity of money supplied by the central bank. Without a Walrasian auctioneer that sets a market-clearing interest rate r_t, there is no way to guarantee that the quantity of money demanded by households will equal the quantity of money supplied by the central bank at time t under simultaneous play if the central bank were to play off of the equilibrium path.

By contrast, in standard games of Bertrand or Cournot competition *within a single industry*, there are no such aggregate constraints that must be respected. Thus, the Walrasian auctioneer is what makes simultaneous play in our macroeconomic game *possible*.

Note that, from the point of view of the game Γ_0, the auctioneer's and economy's market clearing takes place automatically and non-strategically.

2.3 Information Sets and Action Spaces in the Game Γ_0

All firms and households know the structure of the game Γ_0 and the values of all parameters of the game. However, firms and households do not have complete information—in particular, they are *anonymous*, in the sense that their individual actions cannot be observed by other firms or households. Only the history of aggregate variables are publicly observable. This assumption on the information structure of the game simplifies the possible strategies of firms and households that must be considered in determining an equilibrium of the game.

[5] The existence of such a stochastic pricing kernel follows from Hansen and Richard (1987). In the equilibria we consider in this paper, all households will be identical and thus the stochastic pricing kernel will be the usual marginal rate of substitution of the representative household.

Firm i's information set at time t is thus given by $\{L_s, P_s, r_s, w_s, Y_s, \Pi_s, Q_{s,s+1}, p_s(i), y_s(i), l_s(i) : s < t\}$, and household j's information set is given by $\{L_s, P_s, r_s, w_s, Y_s, \Pi_s, Q_{s,s+1}, C_s(j), L_s(j), B_s(j) : s < t\}$. Note that, for each t, $Q_{t,t+1}$ is not a real number but rather a measurable function that describes the time-t price of all state-contingent claims in period $t+1$.

For $i \in [0, 1/2)$, in every even period t firm i submits a price *schedule*, or function, to the Walrasian auctioneer, which can depend on the aggregate variables of the model that are realized at date t: L_t, P_t, r_t, w_t, Y_t, Π_t, and $Q_{t,t+1}$. Thus, rather than a real number, firm i chooses a measurable pricing *function* from Ω into $\mathbb{R}_+$, where $\Omega \equiv \mathbb{R}^6 \times L^1(\mathbb{R})$ is the space of all possible time-t realizations of L_t, P_t, r_t, w_t, Y_t, Π_t, and $Q_{t,t+1}$ (and where $Q_{t,t+1} \in L^1(\mathbb{R})$, the space of Lebesgue-integrable functions on $\mathbb{R}$). Firm i's action space is thus $L(\Omega, \mathbb{R}_+)$, the space of measurable functions from Ω into $\mathbb{R}_+$. In odd periods, firm i has a trivial action space, namely, the continuation of the previous period's price as required by the Taylor contract.

For $i \in [1/2, 1)$, the action spaces of firm i are the same as above, except that the firm submits a price schedule to the auctioneer in odd periods, and has trivial action space in even periods.

For household j in period t, the household likewise submits a joint consumption demand-labor supply schedule to the auctioneer that depends on aggregate variables realized in period t: L_t, P_t, r_t, w_t, Y_t, Π_t, and $Q_{t,t+1}$. The action space of each household j in each period t is thus $L(\Omega, \mathbb{R}^2_+)$, the space of measurable functions from Ω into $\mathbb{R}^2_+$.

Note the diifference between the *action space* for household j, defined above, and a *strategy* for household j. A strategy for household j in game Γ_0 is a sequence of functions $\{\sigma_t(j)\}$, $t \in \mathbb{Z}$, $\sigma_t(j) : \Omega^\omega \to A(j)$, that specify what action household j will play at each time t after observing history $h^t(j)$, where $A(j)$ denotes household j's action space and $h^t(j)$ denotes the history $\{L_s, P_s, r_s, w_s, Y_s, \Pi_s, C_s(j), L_s(j), B_s(j) : s < t\} \in \Omega^\omega$. Thus, even though the household's *action space* at time t is a function only of aggregate variables dated t, the household is still free to follow *strategies* that are functions of the entire history $h^t(j)$ of observed outcomes. That is, the household j is free to play schedules for $C_t(j)$ and $L_t(j)$ that depend on the household's inherited stock of bonds $B_{t-1}(j)$, for example. In fact, it is well known from dynamic programming that the household's optimal strategy depends explicitly on this state variable.

2.4 Payoffs and Optimality Conditions in the Game Γ_0

2.4.1 Firm Payoffs and Optimality Conditions

Each firm i's objective (payoff) in every other period t is to maximize the expected present discounted value of profits over the two periods in which the firm's price remains in effect:

$$E_{it} \left[\Pi_t(i) + Q_{t,t+1} \Pi_{t+1}(i) \right],$$

taking the demand curve for the firm's product in equation (8) as given, where E_{it} denotes the mathematical expectation conditional on firm i's information set at time t, and where $Q_{t,t+1}$ denotes the stochastic discount factor by which the household-owners of the firm value random nominal income at date $t+1$ in monetary units at date t.

Firm optimization implies that, for a firm that is permitted to reset its price in period t, the optimal price $p_t^*(i)$ satisfies the first-order condition:

$$\frac{\partial}{\partial p_t(i)} E_{it}[\Pi_t(i) + Q_{t,t+1}\Pi_{t+1}(i)] = 0,$$

Evaluating this derivative yields the optimality condition:

$$\begin{aligned} p_t^*(i) &= (1+\theta)\, \frac{E_{it} P_t^{(1+\theta)/\theta} Y_t w_t + E_{it} Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{E_{it} P_t^{(1+\theta)/\theta} Y_t + E_{it} Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}}, && (13) \\ &= (1+\theta)\, \frac{P_t^{(1+\theta)/\theta} Y_t w_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{P_t^{(1+\theta)/\theta} Y_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}}, && (14) \end{aligned}$$

where E_t denotes the usual mathematical expectation conditional on all variables dated t and earlier. The second equality follows from the first for two reasons: first, the history of firm i's individual actions does not help to forecast the aggregate variables in (13), because the firm is atomistic and anonymous. Second, even though variables dated t are not yet in firm i's information set at date t when choosing its price schedule, the firm is free to condition its price on aggregate variables dated t by the assumption of the Walrasian auctioneer. The condition (14) is an example of the supply schedule we discussed in section 2.2.2 that the firm submits to the auctioneer. The firm can submit a schedule for $p_t^*(i)$ that is conditioned on all aggregate variables dated t and earlier, as in (14). Equation (14) is just the standard first-order condition from the New Keynesian literature (e.g., Erceg et al., 2000, Woodford, 2003), which we have now derived within the framework of the game Γ_0.

An optimal price $p_t^*(i)$ must also satisfy the second-order condition:

$$\frac{\partial^2}{\partial p_t(i)^2} E_{it}[\Pi_t(i) + Q_{t,t+1}\Pi_{t+1}(i)] \le 0,$$

and ex ante nonnegative expected profit (non-shut-down) condition:

$$E_{it}\left[\Pi_t(i) + Q_{t,t+1}\Pi_{t+1}(i)\right] \ge 0,$$

which yield, respectively:

$$p_t^*(i) \le (1+2\theta)\, \frac{P_t^{(1+\theta)/\theta} Y_t w_t + \beta E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{P_t^{(1+\theta)/\theta} Y_t + \beta E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}},$$

and

$$p_t^*(i) \ge \frac{P_t^{(1+\theta)/\theta} Y_t w_t + \beta E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{P_t^{(1+\theta)/\theta} Y_t + \beta E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}}.$$

Note that, given (14), both inequalities above are satisfied automatically so long as the markup $\theta > 0$, which we will assume throughout.

2.4.2 Household Payoffs and Optimality Conditions

Each household j's payoff in period t is given by the utility flow (3). At each time t, household j faces a standard dynamic programming problem: choose $C_t(j)$ and $L_t(j)$ (and state-contingent plans for the future values of these variables) to maximize the expected present discounted value of payoffs (3) subject to the constraints (4)–(6), taking initial bond holdings $B_{t-1}(j)$ and the stochastic process for $\{L_s, P_s, r_s, w_s, Y_s, \Pi_s, Q_{s,s+1}\}$, $s \geq t$, as given. As was the case for firms, the time-t realizations of the variables L_t, P_t, r_t, w_t, Y_t, Π_t, and $Q_{t,t+1}$ are not yet in the household's information set at time t, but the household is permitted to submit schedules for $C_t(j)$ and $L_t(j)$ to the Walrasian auctioneer that are functions of those time-t realizations.

The solution to this programming problem for the optimal functions $C_t^*(j)$ and $L_t^*(j)$ is well known, though closed-form solutions do not exist in general. Optimal consumption behavior satisfies the Euler equation:

$$C_t^*(j)^{-\varphi} = E_{jt}\beta(1+r_t)\frac{P_t}{P_{t+1}}\, C_{t+1}^*(j)^{-\varphi}, \tag{15}$$

where E_{jt} denotes the mathematical expectation conditional on household j's information set at date t. Optimal labor supply sets the intratemporal marginal rate of substitution equal to the real wage:

$$\chi_0 L_t^*(j)^{\chi} = E_{jt}\frac{w_t}{P_t} C_t^*(j)^{-\varphi}, \tag{16}$$

and using the borrowing limit we can write the budget constraint as:

$$E_{jt}\sum_{T=t}^{\infty} R_{t,T} P_T C_T^*(j) = B_{t-1}(j) + E_{jt}\sum_{T=t}^{\infty} R_{t,T}\left[w_T L_T^*(j) + \Pi_T\right], \tag{17}$$

where $R_{t,T} \equiv \prod_{s=t}^{T}(1+r_s)^{-1}$.

Household j's optimal choice of consumption demand-labor supply schedule at time t, $\{C_t^*(j), L_t^*(j)\}$, is implicitly defined by the three well-known equations (15)–(17).

2.5 Private Sector Equilibrium in the Game Γ_0

We will use the term Private Sector Equilibrium (PSE) to refer to a subgame perfect equilibrium of the game Γ_0:

Definition 1 *Given the i.i.d. stochastic process for* $\{r_t\}$ *and initial conditions* $p_{t_0-1}(i)$ *and* $B_{t_0-1}(j)$ *for all firms* i *and households* j*, we define a Private Sector Equilibrium (PSE) to be a subgame perfect equilibrium of the game* Γ_0*. In particular, a PSE implies a collection of stochastic processes for* $\{L_t, r_t, P_t, w_t, Y_t, \Pi_t, Q_{t,t+1}, p_t(i), y_t(i), l_t(i), C_t(j), L_t(j), B_t(j)\}$ *for* $t \geq t_0$ *and for all* i, j *that satisfy: (i)* *the price optimality condition (14) of the firm's maximization problem; (ii) the consumption and labor optimality conditions (15)–(17) of the household's maximization problem; (iii) the Dixit-Stiglitz aggregation and demand conditions (7)–(9) of the competitive goods aggregator; and (iv) the aggregate resource constraints (10)–(12) imposed by the Walrasian auctioneer.*

This terminology (PSE) becomes useful once we extend our analysis to the game Γ_1 in the next section, where there is an additional player in the form of an optimizing central bank that sets r_t each period. Also note that at this point we have not yet made any restrictions that the strategies played by firms and households be Markov.

2.6 State Variables of the Game Γ_0

It is clear from the definition of PSE above that the game Γ_0 has *two sets* of state variables. First, there is the cross-sectional dispersion of prices $p_{t_0-1}(i)$, $i \in [0,1]$, set by firms in the previous period. Second, there is the continuum of household-specific bond holdings, $B_{t_0-1}(j)$, $j \in [0,1]$, each of which is a state variable for the corresponding household's dynamic programming problem.

In a Markov perfect equilibrium, each player of the game Γ_0 may only follow strategies that depend on the state variables of the game. In Γ_0, firms and households are anonymous, so they only observe the history of aggregate outcomes and their own past actions. Thus, between the two sets of state variables $B_{t-1}(j)$ and $p_{t-1}(i)$, each firm or household may only condition its actions on its own individual value of $B_{t-1}(j)$ or $p_{t-1}(i)$ or on any aggregate quantity that is an observable function of these.

In this section, we show that there are *no* observable aggregate functions of the variables $B_{t-1}(j)$ or $p_{t-1}(i)$ that are relevant for the game Γ_0; that is, there are no aggregate state variables of the game, at least not once we restrict attention (and the definition of the game Γ_0) to the case of a symmetric initial distribution of bond holdings and lagged prices—$B_{t_0-1}(j) = 0$ for all $j \in [0,1]$ and $p_{t_0-1}(i) = p_{t_0-1}$ for some $p_{t_0-1} \in \mathbb{R}_+$, for all $i \in [0,1]$.

2.6.1 Distribution of Bond Holdings

The optimal monetary policy literature is generally not interested in distributional issues, so at this point we restrict the definition of the game Γ_0 to the special case where the initial distribution of bond holdings across households is symmetric—that is, $B_{t_0-1}(j) = 0$ for all $j \in [0,1]$ except possibly a set of measure zero. In that case, all households with $B_{t_0-1}(j) = 0$ will choose the same $\{C_t^*(j), L_t^*(j)\} = \{C_t^*, L_t^*\}$ and the same $B_t^*(j) = B_t^* = 0$ in equilibrium for every time t. We state this as a proposition:

Proposition 1 *Suppose that $B_{t-1}(j)$ is the same for all households $j \in [0,1]$ except possibly a set S of measure zero. Then the optimal action $\{C_t^*(j), L_t^*(j)\} \in L(\Omega, \mathbb{R}_+^2)$ is the same for every household $j \notin S$.*

We denote this optimal action by $\{C_t^*, L_t^*\}$.

Proof. The optimality conditions (15)–(17) for households j_1 and j_2 are identical if $B_{t-1}(j_1) = B_{t-1}(j_2)$. The existence of a unique solution to (15)–(17) is well known when all households have the same initial bond holdings $B_{t-1}(j)$ (e.g., Stokey and Lucas, 1989). Since the set S has measure zero, it has no effect on the existence or uniqueness of this equilibrium. ■

Note that we do not need to assume symmetry here because the actions of other households do not enter into equations (15)–(17); only aggregate variables (and household j's own variables) enter into those equations. Thus, the symmetry of the optimal consumption and labor choices across households (in all periods after the first) is not an assumption but rather an equilibrium implication of the model.

Following Phelan and Stachetti (2001), we will leave unspecified the future strategic behavior of firms and households when a positive measure of households deviate from equilibrium, and simply assert that households and firms continue to play according to the optimality conditions given by (14) and (15)–(17). We will still be able to verify that no household or firm has an incentive to deviate from any equilibrium of the game Γ_0 that we compute.

Finally, it follows from Proposition 1 that the stochastic discount factor by which the economy values future nominal income at date $t+1$ is given (in equilibrium) by the intertemporal marginal rate of substitution of the representative household:

$$Q_{t,t+1}^* = \beta \frac{(C_{t+1}^*)^{-\varphi}}{(C_t^*)^{-\varphi}} \frac{P_t}{P_{t+1}}. \tag{18}$$

2.6.2 Distribution of Prices

We now turn to the cross-sectional distribution of prices. A useful feature of two-period Taylor contracts is that the optimal price set by firms in any period t is always the same across firms, even if the economy inherits dispersed prices from period $t-1$. That is, even if players at time t inherit an out-of-equilibrium value for the distribution of $p_{t-1}(i)$, there is no propagation from that out-of-equilibirum value into the equilibrium at date t or future periods. We state this as a proposition:

Proposition 2 *The optimal choice of price schedule* $p_t^*(i) \in L(\Omega, \mathbb{R}_+)$ *is the same for all firms* i *that reset price in period* t*. We denote this optimal price schedule, given by (14), by* p_t^**.*

Proof. The right-hand side of optimality condition (14) is identical for all firms i. ■

Again, note that symmetry of the optimal price across firms is an equilibrium implication of the model and not an assumption. The actions of other firms do not enter into (14); only aggregate variables enter into that equation.

However, the distribution of lagged prices $p_{t-1}(\cdot)$ typically does have ramifications for the aggregate price level P_t and aggregate output Y_t, through the creation of a distortion in relative prices. The aggregate price level P_t depends on the integral $\int_a^b p_{t-1}(i)^{-1/\theta} di$, where $[a, b) = [0, 1/2)$ for t odd and $[1/2, 1)$ for t even, while aggregate output depends on the integral $\int_a^b p_{t-1}(i)^{-(1+\theta)/\theta} di$ (cf. equation (11)). For a general inherited distribution of lagged prices $p_{t-1}(\cdot)$, it is possible to eliminate one of these two integrals from the game by means of a normalization, but it is not possible to eliminate both of these integrals through a normalization.

As with the cross-sectional distribution of bond holdings, we will thus restrict attention and the definition of the game Γ_0 to the case of a symmetric initial distribution of prices in period t_0, $p_{t-1}(i) = p_{t-1}$ for some p_{t-1} for all firms i, except possibly a set of measure zero. By Proposition 2, it then follows that the cross-sectional distribution of prices $p_t(\cdot)$ in any equilibrium will remain symmetric in every future period.

Thus, we again follow Phelan and Stachetti (2001) and leave unspecified the future strategic behavior of firms and households when a positive measure of firms deviate from equilibrium, and simply assert that households and firms continue to play according to the optimality conditions given by (14) and (15)–(17). We will still be able to verify that no household or firm has an incentive to deviate from any equilibrium of the game Γ_0 that we compute.

2.6.3 Discussion

Given the symmetric initial distribution of bond holdings and last period's prices inherited from period $t_0 - 1$, Propositions 1 and 2 imply that bond holdings and firm prices will remain symmetric in all future periods in any equilibrium of the game Γ_0. If a positive measure of households or firms were to deviate from equilibrium play, then the subgame of Γ_0 from that point forward would have two sets of state variables ($B_{t-1}(j)$ and $p_{t-1}(i)$) that would be relevant for computing any Markov perfect equilibrium of that subgame. As long as a positive measure of households or firms does not deviate, however, then these two distributions are always degenerate and do not provide a state variable on which the players can condition their actions. That is, even if agents in the model do condition their play on these two distributions, $B_{t-1}(j)$ and $p_{t-1}(i)$, Propositions 1 and 2 show that this conditioning will never be operative along any equilibrium path of the game Γ_0. Thus, in computing any and all Markov perfect equilibria of Γ_0, we can effectively ignore these potential state variables, even though they would become relevant if a positive measure of households or firms were to play off of the equilibrium path. This is essentially the argument of Phelan and Stachetti (2001).

2.7 Markov Perfect Equilibrium in the Game Γ_0

We are now ready to define a Markov perfect equilibrium of the game Γ_0. First, let $p_t \equiv 2^{-\theta} \left(\int_a^b p_t(i)^{-1/\theta} di \right)^{-\theta}$ denote the average price set by firms in each period t, where $[a, b) = [0, 1/2)$ for t even and $[1/2, 1)$ for t odd. By Proposition 2 and the discussion above, we know that this distribution $p_t(i)$ is trivial for all t in any Markov perfect equilibrium of Γ_0. We normalize all of the nominal variables in the game at each time t by p_{t-1} and then write down the necessary conditions for an equilibrium of the game that come from the households' and firms' optimality conditions (14) and (15)–(17).

Let x_t denote the normalized average price set by firms in period t:

$$x_t \equiv \frac{p_t}{p_{t-1}}.$$

Then the aggregate price level equation (9) can be written as:

$$\frac{p_t}{P_t} = 2^{-\theta}\big(1 + x_t^{1/\theta}\big)^{\theta}.$$

The aggregate resource constraint for labor (11), in equilibrium, must satisfy:

$$\frac{L_t}{Y_t} = 2^{\theta}\frac{1 + x_t^{(1+\theta)/\theta}}{\big(1 + x_t^{1/\theta}\big)^{1+\theta}}. \tag{19}$$

Using Proposition 2, firms' optimal pricing condition (14) can be rewritten as:

$$2^{-\theta}\big(1 + x_t^{1/\theta}\big)^{\theta} = (1+\theta)\frac{\chi_0\big[Y_t L_t^{\chi} + \beta\big(1 + x_t^{1/\theta}\big)^{1+\theta} h_{1t}\big]}{Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_{2t}}, \tag{20}$$

and households' optimal consumption choice (15) as:

$$Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_{3t}, \tag{21}$$

where h_{1t}, h_{2t}, and h_{3t} denote the expectations:

$$h_{1t} \equiv E_t\frac{Y_{t+1}L_{t+1}^{\chi}}{\big(1 + x_{t+1}^{-1/\theta}\big)^{1+\theta}}, \tag{22}$$

$$h_{2t} \equiv E_t\frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{-1/\theta}}, \tag{23}$$

$$h_{3t} \equiv E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}), \tag{24}$$

and where we replace the household expectation E_{jt} in (24) with the operator E_t by essentially the same argument as in (14): that is, because the household can choose functions that depend on aggregate variables dated t, it is effectively free to condition its expectation operator on all aggregate variables dated t as well as it its information set at date t; moreover, there are no j-specific variables that help forecast the right-hand side of (24), so the operator E_t describes the household's rational expectation of the right-hand side of (24).

Equations (19)–(24) are sufficient to completely determine a private sector equilibrium of the game Γ_0. We state this as a proposition:

Proposition 3 *Given the i.i.d. stochastic process for $\{r_t\}$ and symmetric initial conditions $p_{t_0-1}(i) = p_{t_0-1} \in \mathbb{R}_+$ and $B_{t_0-1}(j) = 0$ for all firms i and households j, necessary and sufficient conditions for a Private Sector Equilibrium are that, for all $t \geq t_0$ and each state of the world: (i) (L_t, x_t, Y_t) satisfy conditions (19)–(21), taking the expectations (h_{1t}, h_{2t}, h_{3t}) as given, and (ii) expectations are rational, so that (h_{1t}, h_{2t}, h_{3t}) are given by (22)–(24).*

Proof. Equations (19)–(24) consolidate the optimality conditions (14) and (15)–(17) and aggregate resource constraints, so the necessity of (19)–(24) is clear. To see the sufficiency, for each period t let $\{C_t(j), L_t(j), B_t(j)\} = \{C_t^*, L_t^*, 0\}$ for all households j, and $p_t(i) = p_t = x_t p_{t-1}$ for all firms i that reset price in period t. Let $C_t^* = Y_t$, $w_t = \chi_0 L_t^{*\chi} C_t^{*\varphi}$, let $y_t(i)$ be given by (8), P_t by (9), $l_t(i) = y_t(i)$, and Π_t by (1) and (2). It is then easy to see that these stochastic processes satisfy the conditions of Definition 1. ■

We now turn to the definition of Markov perfect equilibrium in the game Γ_0:

Definition 2 *A Markov Perfect Equilibrium (MPE) of the game Γ_0 is a set of strategies for households and firms that, at each date t, depend only on the state variables of Γ_0 at time t, and yield a Nash equilibrium in every proper subgame of Γ_0.*

This definition corresponds to the general definition of Markov perfect equilibrium in the literature (e.g., Fudenberg and Tirole, 1993), applied to the special case of our game Γ_0.

Because the state variables of the game Γ_0 are degenerate along the equilibrium path by Propositions 1 and 2, it follows that any Markov perfect equilibrium of Γ_0 between households and firms must involve strategies that are independent of history and time along the equilibrium path:

Proposition 4 *Given the i.i.d. stochastic process for $\{r_t\}$ and symmetric initial conditions $p_{t_0-1}(i) = p_{t_0-1} \in \mathbb{R}_+$ and $B_{t_0-1}(j) = 0$ for all firms i and households j, necessary and sufficient conditions for a Markov Perfect Equilibrium (MPE) of the game Γ_0 are that, for all $t \geq t_0$: (i) (L_t, x_t, Y_t) satisfy households' and firms' optimality conditions (19)-(21), taking r_t and (h_{1t}, h_{2t}, h_{3t}) in as given; (ii) (h_{1t}, h_{2t}, h_{3t}) satisfy conditions (22)-(24) for rational expectations; and (iii) households' and firms' strategies along the equilibrium path are independent of history and independent of time.*

Proof. Propositions 1, 2, and 3, and Definition 2 of Markov perfect equilibrium. ■

The degeneracy of the state variables of Γ_0 along the equilibrium path also implies that the expectations (22)-(24) must be constant along that path, that is:

$$h_{1t} = h_1 \tag{25}$$

$$h_{2t} = h_2 \tag{26}$$

$$h_{3t} = h_3 \tag{27}$$

for some constants h_1, h_2, and h_3, for all times t:

Proposition 5 *Along the equilibrium path of a Markov Perfect Equilibrium of the game Γ_0, there exist positive real numbers h_1, h_2, and h_3 such that $(h_{1t}, h_{2t}, h_{3t}) = (h_1, h_2, h_3)$ for all times t.*

Proof. The variables h_{1t}, h_{2t}, and h_{3t} are mathematical expectations of variables in period $t+1$, conditional on information at date t. Variables in period $t+1$ depend only on variables dated $t+1$ or later, by Proposition 4. The only stochastic variable in Γ_0, r_t, is i.i.d. over time. Thus, the mathematical expectations h_{1t}, h_{2t}, and h_{3t} are the same in every period t. ■

Note that, just because the expectations h_1, h_2, and h_3 are independent of time does not rule out the possibility that there may be multiple equilibria in the game. In particular, there may be multiple sets of expectations (h_1, h_2, h_3), each of which is able to support an MPE in Proposition 4. Alternatively, a given set of expectations (h_1, h_2, h_3) could itself be consistent with multiple MPE in Proposition 4. The remainder of this paper will focus on determining whether there are such multiple MPE in the game Γ_1, defined next.

3 The Optimal Policy Problem

We now turn to the question of the optimal time-consistent policy for $\{r_t\}$ in the above model. We thus wish to add one additional player to the game—an optimizing central bank—and denote this new game by Γ_1. We then calculate the optimal strategies of the central bank and the set of possible Markov perfect equilibria of the new game Γ_1.

3.1 The Central Bank and the Game Γ_1

The central bank differs from households and firms in two key respects. First, the central bank is a large player while households and firms are atomistic—thus, while households and firms take aggregate quantities as given, the central bank understands that its choice of nominal interest rate r_t changes the consumption and labor choices of households and the prices set by firms (through the optimal strategy conditions (15)–(17) and (14)). It is this strategic interaction between interest rate setting and aggregate conditions that is at the heart of central banking.

The second difference is that the central bank faces a dynamic inconsistency between its current plans and its future policy choices. The reason is that the private sector's expectation about the course of future policy has an effect on current economic outcomes, which gives the central bank an incentive to promise a future course for policy today that it may wish to renege on tomorrow.[6] This makes the problem of deriving optimality conditions much more difficult for the case of the central bank than it was for households and firms, and requires us to put considerably more structure on the permissible strategy functions of the central bank than was necessary for households and firms.

[6] In contrast, the private sector faces no such issues in our model: the budget and resource constraints that they face are independent of expectations, so that their problems can be solved by classic dynamic programming.

In the present paper, we are interested in the case of discretion, in which the central bank at time t has no ability to commit to how it will play in future periods $s > t$.

The central bank's information set at time t consists of all aggregate variables dated $t-1$ and earlier: $\{L_s, P_s, r_s, w_s, Y_s, \Pi_s, Q_{s,s+1} : s < t\}$. Like households and firms, the central bank submits an interest rate *schedule* to the Walrasian auctioneer that is a function of all aggregate variables dated t, and the Walrasian auctioneer chooses the equilibrium levels of all variables which satisfy the economy's aggregate resource constraints. Thus, like households and firms, the central bank's action space in the game Γ_1 is the function space $L(\Omega, \mathbb{R})$ of measurable functions from Ω into $\mathbb{R}$. For simplicity, there is no zero lower bound on the central banks's choice of r_t.

The central bank's payoff each period is the average welfare across all households:

$$\int_0^1 \left[\frac{C_T(j)^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_T(j)^{1+\chi}}{1+\chi} \right] dj, \tag{28}$$

which the central bank discounts at the rate β per period. We will derive the central bank's optimality conditions below.

3.2 Motivation for Simultaneous Play in the Game Γ_1

Crucially, we assume that firms, households, and the central bank all play *simultaneously* in each period t in the game Γ_1. By contrast, the previous literature has made the assumption that each period t is divided into two halves, with households and firms playing simultaneously in the second half of each period t, but the central bank being forced to precommit to a value for its monetary policy instrument (either a money stock or an interest rate) in the first half of each period, in a repeated Stackelberg fashion.

We find the assumption of repeated simultaneous play between the central bank and the private sector to be the most natural one for a number of reasons. First, under repeated simultaneous play, it makes no difference whether one defines the monetary instrument to be the short-term nominal interest rate or the money supply—the set of possible equilibria under either assumption for the monetary policy instrument is exactly the same (as we show in section 5). In contrast, this equivalence across monetary instruments does *not* hold under the repeated Stackelberg timing assumption, as shown by Dotsey and Hornstein (2006) and as we discuss in section 5, below.[7] We regard this equivalence between monetary instruments as appealing because most central banks in practice adjust the money supply to maintain a short-term interest rate target, and it is not at all clear which should be regarded as the monetary policy instrument if the two are not equivalent.

[7] Intuitively, adding an additional equation for money demand and changing the monetary authority's instrument to the quantity of money is only guaranteed to yield the same set of equilibrium conditions if the quantity of money and the interest rate are determined simultaneously. Under Stackelberg play, the equivalence between the two instruments is broken because the private sector chooses the interest rate (and output, and expectations, etc.) *after* the monetary authority has already precommitted to a level for the money supply. We will return to this issue in more detail in section 6.

Second, central banks continuously monitor economic conditions and have the freedom and ability to change the monetary instrument continuously, as needed. Thus, the assumption that central banks must precommit to a fixed value of the monetary instrument is at odds with the data—although most central banks find that, along the equilibrium path, they only rarely need to change the instrument between regularly scheduled meetings, this should not be interpreted as a structural constraint on their feasible set of policies or out-of-equilibrium behavior, should they be faced with a sudden deterioration in economic prospects. Although one can address this particular criticism by shrinking the length of a period in the Stackelberg model down to one day or even one hour, it is still not clear why one would want to treat the central bank and the private sector so asymmetrically as to have one or the other always play first. Moreover, shrinking the length of a period does not address the first criticism above.

Third, much existing analysis of optimal time-consistent monetary policy in the New Keynesian model has been done using a linear-quadratic approximation within a simultaneous timing framework (e.g., Svensson and Woodford, 2003, Woodford, 2003, Pearlman, 1992), exactly the timing assumption that we argue should be used in general. Thus, our timing assumption provides the proper generalization of these earlier studies and hence the proper benchmark by which to judge the accuracy or possible misspecification of the LQ approach to optimal monetary policy.

Fourth and finally, the previous literature has typically found multiple equilibria under the optimal monetary policy with discretion. However, these authors' assumption that the monetary authority must precommit to a value for the monetary instrument—and cannot formulate its best response function in terms of the realization of variables at time t—is a substantial constraint on the central bank's ability to control the economy. It is an interesting question, then, whether restoring this control to the central bank (to an extent that is arguably more consistent with the data) would still admit the possibility of multiple equilibria.

3.3 State Variables of the Game Γ_1

The state variables of the game Γ_1 are the same as those of the game Γ_0: in the game Γ_0, the interest rate process $\{r_t\}$ is an exogenous *i.i.d.* process that has no state variables of its own, while in the game Γ_1, the interest rate r_t is set by an optimizing central bank and thus also comes with no state variables of its own, so long as the central bank does not condition its play on an arbitrary past value of a variable—a case which we are explicitly ruling out by restricting attention to Markov perfect equilibria.

As discussed in section 2.6, households' and firms' optimal strategies are independent of all variables in the game Γ_0 dated $t-1$ and earlier. In the game Γ_1, because the interest rate process $\{r_t\}$ likewise has no state variables of its own, the equilibrium conditions (22)–(24) at time t are likewise independent of what has taken place in the past.

3.4 Markov Perfect Equilibrium in the Game Γ_1

The state variables of the game Γ_1 are the same as those of the game Γ_0. As in the game Γ_0, the state variables of Γ_1 are degenerate along the equilibrium path. It is straightforward to see, then, that all of the essential definitions and propositions from section 2 for the game Γ_0 carry over to Γ_1 in the current section as well; in particular, Definition 2 of a Markov perfect equilibrium and Propositions 4 and 5 characterizing Markov perfect equilibrium carry over essentially verbatim to Γ_1. To reiterate, any MPE of Γ_1 between the central bank, households, and firms must involve strategies that are independent of history and time along the equilibrium path, and no matter what interest rate r_t the monetary authority plays in period t, the lack of state variables along the equilibrium path of Γ_1, and the assumption of an MPE implies that the equilibria in period $t+1$ onward are unaffected.

3.5 The Optimal Policy Problem in the Game Γ_1

The optimal policy problem in the game Γ_1, as required by Definition 3, is the optimization of the central bank's objective function (28) subject to the constraints (19)-(21). Following the tradition in the literature, let us define the value function:

$$V_t \equiv \max_{r_t} E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\frac{Y_T^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_T^{1+\chi}}{1+\chi} \right], \tag{29}$$

where the policy function r_t can be chosen as a function of all other variables dated t and earlier (given the existence of the Walrasian auctioneer). The central bank's optimal policy problem thus has the corresponding Bellman equation:

$$V_t = \max_{r_t} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\} \tag{30}$$

where L_t, r_t, and Y_t are related by the three private sector optimality constraints:

$$\frac{L_t}{Y_t} = 2^{\theta} \frac{1 + x_t^{(1+\theta)/\theta}}{\left(1 + x_t^{1/\theta}\right)^{1+\theta}}, \tag{31}$$

$$2^{-\theta}\left(1 + x_t^{1/\theta}\right)^{\theta} \left[Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_{2t}\right] = (1+\theta)\chi_0 \left[Y_t L_t^{\chi} + \beta\left(1 + x_t^{1/\theta}\right)^{1+\theta} h_{1t}\right], \tag{32}$$

$$Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_{3t}. \tag{33}$$

and where expectations are given by the constants:

$$h_{1t} = h_1 = E_t \frac{Y_{t+1} L_{t+1}^{\chi}}{\left(1 + x_{t+1}^{-1/\theta}\right)^{1+\theta}} \tag{34}$$

$$h_{2t} = h_2 = E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{-1/\theta}} \tag{35}$$

$$h_{3t} = h_3 = E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}) \tag{36}$$

In choosing the optimal policy, the central bank takes the constraints (31)–(33) as given, since an MPE of Γ_1 is a Nash equilibrium and thus households and firms must be assumed to play optimally. Moreover, the central bank regards h_1, h_2, and h_3 as exogenous constants, which the central bank cannot affect, and thus h_1, h_2, and h_3 will drop out of the central bank's first-order conditions. Nevertheless, the expectations h_1, h_2, and h_3 can and do influence the central bank's optimal choice of r_t, because they affect the date t realizations of L_t, x_t, and Y_t through the private sector's optimality constraints (31)–(33).

Note also that, because of the absence of state variables along the equilibrium path of Γ_1 and the restriction to MPE, the value function in period $t+1$ does not depend on the action taken by the central bank in period t. Thus, the maximization problem on the right hand side of the Bellman equation (30) is a purely static one, since V_{t+1} is independent of the choice of policy instrument r_t at time t. Maximizing (30) subject to the constraints (31)–(33) then yields the following first-order necessary conditions for the optimal policy:

$$\lambda_t^{\text{Euler}} = 0 \tag{37}$$

$$\chi_0 L_t^{1+\chi} = \lambda_t^Y \frac{L_t}{Y_t} - \lambda_t^x (1+\theta)\chi_0 Y_t \chi L_t^{\chi} \tag{38}$$

$$\lambda_t^Y \frac{L_t}{Y_t} = Y_t^{1-\varphi} + \lambda_t^x \left[(1-\varphi)2^{-\theta}\left(1+x_t^{1/\theta}\right)^{\theta} Y_t^{1-\varphi} - (1+\theta)\chi_0 Y_t L_t^{\chi}\right] \tag{39}$$

$$\lambda_t^Y 2^{\theta} \frac{1+\theta}{\theta} \frac{x_t - 1}{\left(1+x_t^{1/\theta}\right)^{2(1+\theta)}} = \lambda_t^x \left\{ 2^{-\theta} \left[\frac{Y_t^{1-\varphi}}{1+x_t^{1/\theta}} + \frac{1+\theta}{\theta}\beta h_2 \right] - \chi_0 \beta \frac{(1+\theta)^2}{\theta} h_1 \right\} \tag{40}$$

where λ_t^Y, λ_t^x, and λ_t^{Euler} denote the Lagrange multipliers on equations (31), (32), and (33), respectively. Any MPE of the game Γ_1 must satisfy the first-order conditions (31)–(33) and (37)–(40) and the rational expectations conditions (34)–(36).

3.6 The Possibility of Multiple Equilibria: Discussion

The assumption of simultaneous play in period t with a Walrasian auctioneer together with the fact that the central bank is a large, strategic player implies that the central bank has a great deal of control over the private sector variables (L_t, x_t, Y_t) through its choice of policy instrument in period t. Because these variables are determined by a continuum of atomistic agents in the economy, we need the central bank to recognize how the private sector decision rules depend on its policy intrument to exploit this. The assumption that the central bank is a large strategic player here is key, so that it understands that each individual household and firm makes its choices contingent on the central bank's own actions.

Given the power of the central bank to determine aggregate variables at time t through its choice of r_t, one may wonder how multiple equilibria could ever arise in the game Γ_1 in the first place. The possibility of multiple equilibria arises because, although the central bank has a great deal of control over (L_t, x_t, Y_t) and r_t, it has *no* say in what households and firms expect to happen in the next period. Thus, in making its decision, the central bank will take variables h_{1t}, h_{2t}, and h_{3t} as given. A multiplicity can then arise if the central bank's optimal choice of (L_t, x_t, Y_t) and r_t depends on the private sector's expectations about what will happen in the next period. Given different sets of private sector expectations h_1, h_2, and h_3, the game Γ_1 may have different sets of values for (L_t, x_t, Y_t) and r_t that satisfy the conditions for equilibrium.

Note that, just because the expectations h_1, h_2, and h_3 are independent of time does not rule out the possibility that there may be multiple equilibria in the game Γ_1. In particular, there may be multiple sets of expectations (h_1, h_2, h_3), each of which is able to support an MPE as characterized by Proposition 4. Alternatively, a given set of expectations (h_1, h_2, h_3) could itself be consistent with multiple MPE under Proposition 4. Whether or not this can occur is a question we need to ask the model.

4 Solution for Markov Perfect Equilibria in the Game Γ_1

The restriction of attention to Markov Perfect Equilibria and the absence of state variables along the equilibrium path of Γ_1 combine to make a closed-form solution for the optimal policy and associated equilibria feasible. Since interest often centers around the economy's inflation rate $\pi_t \equiv P_t/P_{t-1}$, we will also define that auxiliary variable here. Note that:

$$\pi_t^{1/\theta} = \frac{1 + x_{t-1}^{1/\theta}}{1 + x_t^{-1/\theta}} \tag{41}$$

The following proposition clarifies that all possible MPE of the game Γ_1 have a very simple structure over time:

Proposition 6 *Let* $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1})$ *and* $(L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$ *lie on the equilibrium path of an MPE of* Γ_1*. Then* $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1}) = (L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$*. That is, along the equilibrium path, any MPE of* Γ_1 *must be constant over time.*

Proof. By Proposition 4, households, firms, and the central bank all must play strategies that are independent of history and of time. Given that expectations h_1, h_2, and h_3 are also independent of time by Proposition 5, any MPE that satisfies (31)–(40) is independent of time. ■

As a result of Proposition 6, we know that any MPE of Γ_1 is constant over time. The central question, then, is if the definition of the game Γ_1 allows for more than one such equilibrium. If there are multiple equilibria that satisfy Proposition 6 above, then there would be reason to go further and try to specify which equilibrium would be chosen in each period t, e.g., through the introduction of an exogenous, publicly observable "sunspot" variable that could act as a coordinating device for agents of the model. (However, it turns out that this will not be necessary, due to Proposition 7, below.)

It follows from Proposition 6 that, along the equilibrium path of any MPE of Γ_1, the expectation of all variables dated $t + 1$ is the same as the values of those variables in period t, which simplifies equations

(31)–(40) substantially. After some algebraic manipulation, it can be shown that the optimal policy problem (31)–(40) can be reduced down to a single equation for π, as follows:

Proposition 7 *The inflation rate π in any Markov Perfect Equilibrium of the game Γ_1 must satisfy the condition:*

$$\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\,\frac{1+\pi^{1/\theta}}{1+\pi^{(1+\theta)/\theta}}\left\{1-\frac{(\pi-1)\left[1+\chi-(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]}{(\pi-1)\left[1-(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]+(1+\pi^{(1+\theta)/\theta})\left[1-\frac{1}{1+\theta}\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]}\right\}=\frac{1}{1+\theta} \tag{42}$$

Proof. The proof is just algebraic consolidation of the first-order necessary conditions (31)–(33), (34)–(36), and (37)–(40), after applying Proposition 7. See the Appendix for details. ■

The central question of this paper is then, for what sets of parameter values β, θ, φ, and χ does there exist more than one possible equilibrium value of π? In the Appendix, we prove the following proposition:

Proposition 8 *Let $\varphi = 1$, $\chi = 0$, and $\beta > \max\{1/2, 1/(1+2\theta)\}$. Then there is precisely one value of π that satisfies equation (42).*

Proof. See Appendix. ■

In the Proposition and proof above, there is nothing crucial about the parameter values $\varphi = 1$, $\chi = 0$, they merely simplify the algebra, and the number of cases that must be considered in the proof, considerably. Indeed, Proposition 7 can be generalized to a much wider class of parameter values for φ and χ, but the proof becomes much more algebraically convoluted.

As an alternative and a check on the algebraic approach in the proof of Proposition 7, we have conducted extensive numerical computations in Matlab which compute all solutions to the equation (42) for a given set of values for the parameters β, θ, φ, and χ. The code underlying these computations is available from our web sites, and shows that there is a unique equilibrium value of π for a wide range of parameter values, including [specify range here].

5 The Introduction of Money

In our previous discussion we assumed that the nominal interest rate is the instrument of policy. It will not change the result if we assume instead that the money supply is the instrument of policy. To see this, suppose that there is a cash advance constraint as in King and Wolman (2004), so that:

$$\frac{M_t}{P_t} = Y_t, \tag{43}$$

in each period. We can rewrite this in terms of the variables of the previous section as:

$$m_t x_t^{-1} 2^{-\theta} (1 + x_t^{1/\theta})^{\theta} = Y_t, \tag{44}$$

where we have defined $m_t \equiv \frac{M_t}{p^*_{t-1}}$. Consider now a policy in which m_t is the policy instrument instead of r_t. In this case the Lagrangian formulation in Section 3.4. is exactly the same except we now take equation (44) into account. Denoting the Lagrangian multiplier of this constraint by λ_t^{Money} we obtain once again the same conditions as before because the first order condition with respect to m_t indicates that

$$\lambda_t^{Money} = 0. \tag{45}$$

Hence introducing money explicitly into the analysis under our timing assumption has no effect on the results derived in the last few sections.

6 Discussion of Stackelberg vs. Simultaneous Play

As an alternative to the game Γ_1, in which there is repeated simultaneous play by all agents in every period, one can instead consider the alternative timing assumption in which play between the central bank and the private sector is repeated Stackelberg. Under this alternative within-period timing assumption, the central bank moves before households and firms and precommits to a particular value of the money supply (or interest rate). This is the approach taken, for example, in King and Wolman (2004).

For the purposes of this section, then, consider a new game, Γ_2, which differes from Γ_1 in that each period is now divided into two halves. In the first half of each period, the central bank chooses a money supply $m_t \in \mathbb{R}_+$, that is, the central bank chooses a *fixed* value for the money supply and "precommits" to that value for the remainder of the period.

Firms and households are defined as in the game Γ_0 in Section 2. However, in the game Γ_2, households and firms play in the second half of each period, after the central bank has announced and committed to its value for the money supply. As in the game Γ_0, however, firms and households play simultaneously with each other, choosing price *schedules* and joint labor supply-consumption demand *schedules* that are functions of the relevant aggregate variables realized at time t, such as the wage rate, price level, and aggregate level of output. As in the game Γ_0, the Walrasian auctioneer determines the values of these aggregate variables that clear each of the labor, goods, and bond markets. Note, however, that in the game Γ_2, firms and households already know the value of the central bank's money supply m_t, because that has been announced in the first half of the period.

Thus, in the game Γ_2, firms and households have a state variable, namely the money supply set by the central bank in the first half of the period. This is in contrast to the games Γ_0 and Γ_1, in which firms and households had no state variable. The restriction to a Markov Perfect Equilibrium, then, requires that firms and households can only condition their actions on the "minimum state", which in this case is the money supply. Thus, we can write the decision rules of the private sector as:

$$Y_t = \bar{Y}(m_t),\ x_t = \bar{x}(m_t),\ L_t = \bar{L}(m_t),\ r_t = \bar{r}(m_t) \tag{46}$$

which the central bank takes as given in selecting its policy m_t. At the beginning of each period, then, the government then solves:

$$V_t = \max_{m_t} \left[\frac{Y_T^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_T^{1+\chi}}{1+\chi} \right] + \beta E_t V_{t+1} \tag{47}$$

subject to (31)-(33), (44), and (46).

Note that under the repeated Stackelberg within-period timing assumption, the central bank has much less control over the values of Y_t, r_t, and L_t with its choice of action, in this case the level of the money supply.

Note that the government will take the expectations in period t as some given constants, because they are based on expectations about things that happen in period $t+1$ and thus independent of the current policy choice. Observe the difference in the definition of the equilibrium under the game Γ_2 (repeated Stackelberg timing) versus the game Γ_1 (repeated simultaneous timing). Because the central bank chooses a value for m_t before firms and households act, this in itself can give rise to multiplicity because there is no guarantee that (46) have a unique solution for a given value of m_t. Indeed, as King and Wolman (2004) show, there are in general two solutions for (L_t, x_t, Y_t) for any one value of m_t. It is therefore the multiplicity of the functions $\bar{Y}(m_t)$, $\bar{x}(m_t)$, and $\bar{L}(m_t)$ that is the source of multiplicity in the game Γ_2.

It is interesting to note that under the repeated Stackelberg timing assumption of the game Γ_2, whether the central bank's policy instrument is the money supply m_t or the interest rate r_t is important. When the central bank precommits to a value for the nominal interest rate r_t at the beginning of the period, instead of the money supply, we replace (46) by:

$$Y_t = \bar{Y}(r_t),\ x_t = \bar{x}(r_t),\ L_t = \bar{L}(r_t),\ m_t = \bar{m}(r_t) \tag{48}$$

We can then once again define an MPE as in the last defintion, but replacing the m_t with r_t and (46) with (48). As Dotsey and Hornstein and (2006) show, there is a unique equilibrium in this case.

7 Conclusions

In this paper, we have emphasized that there are two separate approaches that have been followed in the literature on optimal time-consistent monetary policy. These two approaches differ with respect to their assumptions regarding the within-period timing of play, which we have called “repeated simultaneous” and “repeated Stackelberg” play. We argue that this distinction and its importance have not been fully appreciated in the previous literature. In particular, we show that the within-period timing of play can be critical for determining whether or not there are multiple Markov perfect equilibria in a standard New Keynesian monetary model. We illustrate this by working through the details of the two-period New Keynesian model of King and Wolman (2004) under the assumption that within-period play between the central bank, households, and firms is repeated simultaneous rather than repeated Stackelberg, and show that there is a unique Markov perfect equilibrium under that timing assumption. It remains to be seen if this insight generalizes to other models.

Appendix: Proofs of Propositions

Proposition 7 *The inflation rate* π *in any Markov Perfect Equilibrium of the game* Γ_1 *must satisfy the condition:*

$$\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\,\frac{1+\pi^{1/\theta}}{1+\pi^{(1+\theta)/\theta}}\left\{1-\frac{(\pi-1)\left[1+\chi-(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]}{(\pi-1)\left[1-(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]+(1+\pi^{(1+\theta)/\theta})\left[1-\frac{1}{1+\theta}\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]}\right\}=\frac{1}{1+\theta}, \tag{49}$$

Proof. The proof is just algebraic consolidation of the first-order necessary conditions (31)–(33), (34)–(36), and (37)–(40). Plugging (34)–(36) into (31)–(33) and (37)–(40), simplifying, and applying Proposition 5 yields:

$$\frac{L}{Y}=2^{\theta}\frac{1+x^{(1+\theta)/\theta}}{\left(1+x^{1/\theta}\right)^{1+\theta}}, \tag{50}$$

$$2^{-\theta}\left(1+x^{1/\theta}\right)^{\theta}Y^{1-\varphi}\left[1+\beta x^{1/\theta}\right]=(1+\theta)\chi_0 YL^{\chi}\left[1+\beta x^{(1+\theta)/\theta}\right], \tag{51}$$

$$x^{1/\theta}=\beta(1+r). \tag{52}$$

$$\chi_0 L^{1+\chi}=\lambda^{Y}\frac{L}{Y}-\lambda^{x}(1+\theta)\chi_0 Y\chi L^{\chi}, \tag{53}$$

$$\lambda^{Y}\frac{L}{Y}=Y^{1-\varphi}+\lambda^{x}\left[(1-\varphi)2^{-\theta}\left(1+x^{1/\theta}\right)^{\theta}Y^{1-\varphi}-(1+\theta)\chi_0 YL^{\chi}\right], \tag{54}$$

$$\lambda^{Y}2^{\theta}\frac{1+\theta}{\theta}\,\frac{x-1}{\left(1+x^{1/\theta}\right)^{(1+\theta)}}=\lambda^{x}\left\{2^{-\theta}Y^{1-\varphi}\left(1+x^{1/\theta}\right)^{\theta}\left[1+\beta\frac{1+\theta}{\theta}x^{1/\theta}\right]-\beta\frac{(1+\theta)^2}{\theta}\chi_0 YL^{\chi}x^{(1+\theta)/\theta}\right\}. \tag{55}$$

Substituting, simplifying, and recognizing $x=\pi$ from (41) yields:

$$\frac{L}{Y}=2^{\theta}\frac{1+\pi^{(1+\theta)/\theta}}{\left(1+\pi^{1/\theta}\right)^{1+\theta}}, \tag{56}$$

$$2^{-\theta}\left(1+\pi^{1/\theta}\right)^{\theta}Y^{1-\varphi}=(1+\theta)\chi_0 YL^{\chi}\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}, \tag{57}$$

$$\chi_0 L^{1+\chi}-Y^{1-\varphi}=\lambda^{x}(1+\theta)\chi_0 YL^{\chi}\left[(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}-(1+\chi)\right], \tag{58}$$

$$\begin{aligned}&\left[\lambda^{x}(1+\theta)\chi_0 Y\chi L^{\chi}+\chi_0 L^{1+\chi}\right]\frac{1+\theta}{\theta}\,\frac{\pi-1}{1+\pi^{(1+\theta)/\theta}}=\\ &\lambda^{x}(1+\theta)\chi_0 YL^{\chi}\left\{\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}+\beta\frac{1+\theta}{\theta}\pi^{1/\theta}\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}-\beta\frac{1+\theta}{\theta}\pi^{(1+\theta)/\theta}\right\},\end{aligned} \tag{59}$$

which can be simplified further to yield:

$$(1+\theta)\chi_0 L^{1+\chi}\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}=Y^{1-\varphi}\frac{1+\pi^{(1+\theta)/\theta}}{1+\pi^{1/\theta}}, \tag{60}$$

$$\left\{\chi_0 L^{1+\chi}\left[(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}-1\right]-\chi Y^{1-\varphi}\right\}\frac{1+\theta}{\theta}\,\frac{\pi-1}{1+\pi^{(1+\theta)/\theta}}=$$

$$\left(\chi_0 L^{1+\chi}-Y^{1-\varphi}\right)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\left\{1+\beta\frac{1+\theta}{\theta}\pi^{1/\theta}\frac{1-\pi}{1+\beta\pi^{(1+\theta)/\theta}}\right\}, \tag{61}$$

which together yield:

$$\left\{(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}-1-\chi(1+\theta)\,\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\,\frac{1+\pi^{1/\theta}}{1+\pi^{(1+\theta)/\theta}}\right\}\frac{\pi-1}{1+\pi^{(1+\theta)/\theta}}=$$

$$\left(1-(1+\theta)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\,\frac{1+\pi^{1/\theta}}{1+\pi^{(1+\theta)/\theta}}\right)\left\{1-\frac{1}{1+\theta}\,\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right\}, \tag{62}$$

and:

$$(1+\theta)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\,\frac{1+\pi^{1/\theta}}{1+\pi^{(1+\theta)/\theta}}\left\{(1+\pi^{(1+\theta)/\theta})\left[1-\frac{1}{1+\theta}\,\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]-(\pi-1)\chi\right\}=$$

$$(\pi-1)\left[1-(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]+(1+\pi^{(1+\theta)/\theta})\left[1-\frac{1}{1+\theta}\,\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]. \tag{63}$$

A final rearrangement of terms yields (49). ■

To simplify notation in the remainder of the appendix, define the quantities:

$$\begin{aligned}
\beta^{rat} &\equiv \frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}},\\
\pi^{rat} &\equiv \frac{1+\pi^{1/\theta}}{1+\pi^{(1+\theta)/\theta}},\\
N &\equiv (\pi-1)\left[1+\chi-(1-\varphi)\beta^{rat}\right],\\
D &\equiv (\pi-1)[1-(1-\varphi)\beta^{rat}]+(1+\pi^{(1+\theta)/\theta})\left[1-\frac{1}{1+\theta}\beta^{rat}\right],
\end{aligned}$$

We also define the function:

$$F(\pi)\equiv(1+\theta)\beta^{rat}\pi^{rat}\left\{1\;-\;\frac{N}{D}\right\}. \tag{64}$$

Any equilibrium of the New Keynesian model must satisfy equation (49) or, equivalently:

$$F(\pi)=1. \tag{65}$$

To simplify notation and the proofs further, we restrict attention to the special case $\varphi=0$ and $\chi=1$. This simplifies N and D to:

$$\begin{aligned}
N &\equiv (\pi-1),\\
D &\equiv (\pi-1)+(1+\pi^{(1+\theta)/\theta})\left[1-\frac{1}{1+\theta}\beta^{rat}\right].
\end{aligned}$$

The following lemma collects some observations that will be useful in the proofs that follow.

Lemma 9

$$\beta^{rat} = 1 + \frac{\beta\pi^{1/\theta}}{1+\beta\pi^{1/\theta}}(\pi - 1), \tag{66}$$

$$\beta^{rat}\pi^{rat} = 1 - \frac{(1-\beta)\pi^{1/\theta}(\pi-1)}{(1+\beta\pi^{1/\theta})(1+\pi^{(1+\theta)/\theta})}, \tag{67}$$

$$\frac{\pi}{\beta^{rat}} = 1 + \frac{\pi - 1}{1+\beta\pi^{(1+\theta)/\theta}}, \tag{68}$$

$$\frac{d\beta^{rat}}{d\pi} = \frac{\beta^{rat}}{\pi}\frac{\beta\pi^{1/\theta}}{1+\beta\pi^{1/\theta}}\left[\frac{1+\theta}{\theta}\frac{\pi}{\beta^{rat}} - \frac{1}{\theta}\right]. \tag{69}$$

Proof. Elementary. ■

Lemma 10 *The function $D(\pi)$ defined above takes on the value zero for precisely one value of π in the interval $(0,1)$ and for precisely one value of π in the interval $(1,\infty)$.*

Proof. Note first that $D(\pi)$ is continuous (and differentiable) for all $\pi \geq 0$ and that $D(0) = -1/(1+\theta)$, $D(1) = 2\theta/(1+\theta)$, and $\lim_{\pi\to\infty} D(\pi) = -\infty$. Thus D has at least one zero in $(0,1)$ and at least one zero in $(1,\infty)$.

1) Consider first the case $\pi \geq 1$. We will prove that once $D \leq 0$, then $dD/d\pi < 0$ and thus D can never have another zero. Differentiating gives:

$$\frac{dD}{d\pi} = 1 + \frac{1+\theta}{\theta}\pi^{1/\theta} - \frac{1}{\theta}\pi^{1/\theta}\beta^{rat} + \frac{D}{\beta^{rat}}\frac{d\beta^{rat}}{d\pi} - \frac{\pi(1+\pi^{1/\theta})}{\beta^{rat}}\frac{d\beta^{rat}}{d\pi}. \tag{70}$$

Rearrange and combine terms in (70) to get:

$$\frac{D}{\beta^{rat}}\frac{d\beta^{rat}}{d\pi} - \frac{\pi(1+\pi^{1/\theta})}{\beta^{rat}}\frac{d\beta^{rat}}{d\pi} + \frac{1+\theta}{\theta}\frac{D}{\pi} - \frac{1}{\theta}\left(1 - \frac{\beta^{rat}}{\pi}\right), \tag{71}$$

and note that for $\pi \geq 1$ and $D \leq 0$, every term in the expression above is negative or zero.

2) Now consider the case $\pi \leq 1$. We will prove that if $D \geq 0$, then $dD/d\pi > 0$ and thus D can have at most one zero in $(0,1)$.

Rearrange terms in (70) and substitute in for $d\beta^{rat}/d\pi$ to get:

$$\frac{1}{\theta}\pi^{1/\theta}(1-\beta^{rat}) + (1+\pi^{1/\theta})\left[\frac{1}{1+\beta\pi^{(1+\theta)/\theta}} + \frac{1}{\theta}\frac{\beta\pi^{1/\theta}}{1+\beta\pi^{1/\theta}}\left(1 - \frac{\pi}{\beta^{rat}}\right)\right] + \frac{D}{\beta^{rat}}\frac{d\beta^{rat}}{d\pi}, \tag{72}$$

and if $D \geq 0$, then every term above is positive or zero (noting that $\pi < \beta^{rat} < 1$ by (68) and (66)), completing the proof. ■

Proposition 11 *There is no solution to (65) with $\pi \leq 1$.*

Proof. Let π_0 denote the zero of D in the interval $(0,1)$. We first show that there is no solution to (65) with $\pi \leq \pi_0$. For $\pi \leq \pi_0$, we have both $D \leq 0$ and:

$$D - N = (1+\pi^{(1+\theta)/\theta})\left(1 - \frac{1}{1+\theta}\beta^{rat}\right) > 0, \tag{73}$$

which implies that $N < D \leq 0$ and hence $F(\pi) < 0$, which cannot satisfy (65).

For $\pi \in (\pi_0, 1]$, we have $D > 0$ and $N < 0$, and hence $(1 - N/D) > 1$. But by (67) we also have $\beta^{rat}\pi^{rat} \geq 1$, which implies $F(\pi) > 1$, completing the proof. ■

Proposition 12 *Let π_1 denote the zero of D in the interval $(1, \infty)$. For $\beta > 1/(1+2\theta)$, there is no solution to (65) with $\pi \geq \pi_1$.*

Proof. Recall that $\theta > 0$ denotes the monopolistic markup. Thus, the proposition states that, so long as the social discount factor β and the markup θ are not extraordinarily low, there can be no equilibrium with $\pi \geq \pi_1$.

For $\pi \geq \pi_1$, we have $D < 0$ and $N > 0$, hence $(1 - N/D) > 1$. We will show that $\beta^{rat}\pi^{rat} > 1/(1 + \theta)$ under the stated condition on β, in which case $F(\pi) > 1$ and there can be no equilibrium. From (67), we have:

$$\begin{aligned}
\beta^{rat}\pi^{rat} &= 1 - \frac{(1-\beta)\pi^{1/\theta}(\pi - 1)}{(1+\beta\pi^{1/\theta})(1+\pi^{(1+\theta)/\theta})}, \\
&> 1 - \frac{(1-\beta)\pi^{(1+\theta)/\theta}}{(1+\beta\pi^{1/\theta})(1+\pi^{(1+\theta)/\theta})}, \\
&> 1 - \frac{(1-\beta)}{(1+\beta\pi^{1/\theta})}, \\
&> 1 - \frac{1-\beta}{1+\beta}, \\
&= \frac{2\beta}{1+\beta}, \\
&> \frac{1}{1+\theta,}
\end{aligned}$$

where the last inequality follows from the assumed condition on β. ■

Proposition 13 *Let π_1 denote the zero of D in the interval $(1, \infty)$. For $\beta > 1/2$, there is precisely one solution to (65) with $\pi \in (1, \pi_1)$.*

Proof. Note that $F(1) = 1 + \theta$ and $F(\pi_1) = -\infty$, and F is continuous (and differentiable) on $(1, \pi_1)$, so there is at least one solution to (65) in $(1, \pi_1)$.

Let π_2 denote the solution to $\beta^{rat} = 1 + \theta$. Note that $\beta^{rat}(1) = 1$, β^{rat} is increasing in $(1, \pi_1)$ by (69), and $\beta^{rat}(\pi_1) > 1 + \theta$, where the last inequality follows from the fact that $D(\pi_1) = 0$. So one and only one such solution π_2 exists in $(1, \pi_1)$. Now, for $\pi \geq \pi_2$, we have $\beta^{rat} \geq 1 + \theta$, hence $N \geq D$ and $F(\pi) \leq 0$. Thus, there is no solution to (65) with $\pi \geq \pi_2$.

We complete the proof of the proposition by showing that $F' < 0$ on $(1, \pi_2)$.

We have

$$\frac{dF}{d\pi} = (1+\theta)\left(1-\frac{N}{D}\right)\left[\frac{d\beta^{rat}\pi^{rat}}{d\pi}\right] + (1+\theta)\left[N\frac{dD}{d\pi} - D\right], \tag{74}$$

and recall that $(1 - N/D) > 0$ on $(1, \pi_2)$.

Now,

$$\begin{aligned}
\frac{d\beta^{rat}\pi^{rat}}{d\pi} &= \frac{(1-\beta)\pi^{(1-\theta)/\theta}(\pi-1)}{(1+\beta\pi^{1/\theta})(1+\pi^{(1+\theta)/\theta})}\left[\frac{-\pi}{\pi-1} - \frac{1}{\theta} + \frac{1}{\theta}\frac{\beta\pi^{1/\theta}}{1+\beta\pi^{1/\theta}} + \frac{1+\theta}{\theta}\frac{\pi^{(1+\theta)/\theta}}{1+\pi^{(1+\theta)/\theta}}\right], \\
&= \frac{(1-\beta)\pi^{(1-\theta)/\theta}(\pi-1)}{(1+\beta\pi^{1/\theta})(1+\pi^{(1+\theta)/\theta})}\left[\frac{-1}{\pi-1} + \frac{1}{\theta}\frac{\beta\pi^{1/\theta}}{1+\beta\pi^{1/\theta}} - \frac{1+\theta}{\theta}\frac{1}{1+\pi^{(1+\theta)/\theta}}\right], \\
&= \frac{(1-\beta)\pi^{(1-\theta)/\theta}}{(1+\beta\pi^{1/\theta})(1+\pi^{(1+\theta)/\theta})}\left[\frac{1}{\theta}(\beta^{rat} - 1 - \theta) - \frac{1+\theta}{\theta}\frac{\pi-1}{1+\pi^{(1+\theta)/\theta}}\right],
\end{aligned}$$

which is less than zero on $(1, \pi_2)$ because $\beta^{rat} < 1+\theta$ on that interval.

Moreover,

$$\begin{aligned}
N\frac{dD}{d\pi} - D &= (\pi-1)\left[\frac{1+\theta}{\theta}\pi^{1/\theta}\left(1-\frac{\beta^{rat}}{1+\theta}\right) - \frac{1}{1+\theta}(1+\pi^{(1+\theta)/\theta})\frac{d\beta^{rat}}{d\pi}\right] - (1+\pi^{(1+\theta)/\theta})\left(1-\frac{\beta^{rat}}{1+\theta}\right), \\
&= \left(1-\frac{\beta^{rat}}{1+\theta}\right)\left[\frac{1+\theta}{\theta}(\pi^{(1+\theta)/\theta} - \pi^{1/\theta}) - (1+\pi^{(1+\theta)/\theta})\right] - \frac{1+\pi^{(1+\theta)/\theta}}{1+\theta}\frac{(\pi-1)\beta\pi^{1/\theta}}{1+\beta\pi^{1/\theta}}\left[\frac{1+\theta}{\theta} - \frac{1}{\theta}\frac{\beta^{rat}}{\pi}\right], \\
&= \left(1-\frac{\beta^{rat}}{1+\theta}\right)\left[\frac{1}{\theta}\pi^{(1+\theta)/\theta} - \frac{1+\theta}{\theta}\pi^{1/\theta} - 1\right] - \frac{1}{1+\theta}(1+\pi^{(1+\theta)/\theta})(\beta^{rat}-1)\left[1+\frac{1}{\theta}\left(1-\frac{\beta^{rat}}{\pi}\right)\right].
\end{aligned}$$

If $\left[\frac{1}{\theta}\pi^{(1+\theta)/\theta} - \frac{1+\theta}{\theta}\pi^{1/\theta} - 1\right] \le 0$, then $N\frac{dD}{d\pi} - D < 0$ and $F' < 0$, as was to be shown.

If instead $\left[\frac{1}{\theta}\pi^{(1+\theta)/\theta} - \frac{1+\theta}{\theta}\pi^{1/\theta} - 1\right] > 0$, then we have $\beta\pi^{(1+\theta)/\theta} > \beta\theta + \beta(1+\theta)\pi^{1/\theta}$ and hence $\beta^{rat} > 1+\theta - \frac{\theta(1-\beta)}{1+\beta\pi^{1/\theta}}$. Recall also that $\beta^{rat} < 1+\theta$ on $(1, \pi_2)$, which implies that $\left[\frac{1}{\theta}\pi^{(1+\theta)/\theta} - \frac{1+\theta}{\theta}\pi^{1/\theta} - 1\right] < (1-\beta)/\beta$. Applying these inequalities to the expression for $N\frac{dD}{d\pi} - D$ yields:

$$\begin{aligned}
N\frac{dD}{d\pi} - D &< \left(\frac{\theta}{1+\theta}\frac{1-\beta}{1+\beta\pi^{1/\theta}}\right)\left[\frac{1-\beta}{\beta}\right] - \frac{1}{1+\theta}\frac{1}{\pi^{rat}}\frac{\theta\beta}{1+\beta\pi^{1/\theta}}\left[1+\frac{1}{\theta}\left(1-\frac{\beta^{rat}}{\pi}\right)\right], \\
&= \frac{\theta}{1+\theta}\frac{1}{1+\beta\pi^{1/\theta}}\left[\frac{(1-\beta)^2}{\beta} - \frac{\beta}{\pi^{rat}}\left(1+\frac{1}{\theta}\left(1-\frac{\beta^{rat}}{\pi}\right)\right)\right], \\
&< \frac{\theta}{1+\theta}\frac{\beta}{1+\beta\pi^{1/\theta}}\left[\frac{(1-\beta)^2}{\beta^2} - 1\right],
\end{aligned}$$

which is less than zero under the stated assumption that $\beta > 1/2$, completing the proof. ■

Proposition 8 *Let $\varphi = 1$, $\chi = 0$, and $\beta > \max\{1/2, 1/(1+2\theta)\}$. Then there is precisely one value of π that satisfies equation (49).*

Proof. The proposition follows immediately from the Lemmas and Propositions above. ■

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