Risk Aversion, Risk Premia, and the Labor Margin with Generalized Recursive Preferences

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Coefficient of Relative Risk Aversion

Suppose a household has preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \, l_t$$

What is the household's coefficient of relative risk aversion?

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Answer: 0

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What is the household's coefficient of relative risk aversion?

Answer: $\dfrac{1}{\frac{1}{\gamma} + \frac{1}{\chi}}$

Empirical Relevance of the Labor Margin

Imbens, Rubin, and Sacerdote (2001):

- Individuals who win a lottery prize reduce labor supply by $.11 for every $1 won (note: spouse may also reduce labor supply)

Coile and Levine (2009):

- Older individuals are 7% less likely to retire in a given year after a 30% fall in stock market

Coronado and Perozek (2003):

- Individuals who held more stocks in late 1990s retired 7 months earlier

Large literature estimating wealth effects on labor supply (e.g., Pencavel 1986)

Household with Generalized Recursive Preferences

Household chooses state-contingent $\{(c^t, l^t)\}$ to maximize

$$V(a_t;\theta_t) = \max_{(c_t,l_t)\in\Gamma(a_t;\theta_t)} u(c_t, l_t) + \beta \left(E_t\, V(a_{t+1};\theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)}$$

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Note: Generalized recursive preferences are often written as:

$$U(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} \left[\widetilde{u}(c_t, l_t)^{\rho} + \beta \left(E_t \, U(a_{t+1}; \theta_{t+1})^{\widetilde{\alpha}} \right)^{\rho/\widetilde{\alpha}} \right]^{1/\rho}$$

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It's easy to map back and forth from U to V; moreover,

- V is more closely related to standard dynamic programming results, regularity conditions, and FOCs
- V makes derivations, formulas in the paper simpler
- additively separable u is easier to consider in V

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subject to flow budget constraint

$$a_{\tau+1} = (1 + r_\tau) a_\tau + w_\tau l_\tau + d_\tau - c_\tau$$

and No-Ponzi condition.

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State variables of the household's problem are $(a_t;\theta_t)$.

Let:

$$c_t^* \equiv c^*(a_t;\theta_t),$$

$$l_t^* \equiv l^*(a_t;\theta_t).$$

Technical Conditions

Assumption 1. *The function $u(c_t, l_t)$ is increasing in its first argument, decreasing in its second, twice-differentiable, and strictly concave.*

Assumption 2. *Either $u\colon \Omega \to [0, \infty)$ or $u\colon \Omega \to (-\infty, 0]$.*

Assumption 3. *A solution $V\colon X \to \mathbb{R}$ to the household's generalized Bellman equation exists and is unique, continuous, and concave.*

Assumption 4. *For any $(a_t; \theta_t) \in X$, the household's optimal choice (c_t^*, l_t^*) exists, is unique, and lies in the interior of $\Gamma(a_t; \theta_t)$.*

Assumption 5. *For any $(a_t; \theta_t)$ in the interior of X, the second derivative of V with respect to its first argument, $V_{11}(a_t; \theta_t)$, exists.*

Assumptions about the Economic Environment

Assumption 6. *The household is* infinitesimal*.*

Assumption 7. *The household is* representative*.*

Assumption 8. *The model has a* nonstochastic steady state, $x_t = x_{t+k}$ *for* $k = 1, 2, \ldots,$ *and* $x \in \{c, l, a, w, r, d, \theta\}$.

Assumption 8′. *The model has a* balanced growth path *that can be renormalized to a nonstochastic steady state after a suitable change of variables.*

Arrow-Pratt in a Static One-Good Model

Compare:

$$E\,u(c + \sigma\varepsilon) \quad \text{vs.} \quad u(c - \mu)$$

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Arrow-Pratt coefficient of absolute risk aversion:

$$\lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2$$

Arrow-Pratt in a Dynamic Model

Consider a one-shot gamble in period t:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + \sigma\varepsilon_{t+1},$$

vs.

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu.$$

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Definition 1. *The household's coefficient of absolute risk aversion at $(a_t; \theta_t)$ is given by* $R^a(a_t; \theta_t) = \lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2$.

Coefficient of Absolute Risk Aversion

Proposition 1. *The household's coefficient of absolute risk aversion at* $(a_t; \theta_t)$*, denoted* $R^a(a_t; \theta_t)$*, satisfies*

$$\frac{-E_t\left[V(a^*_{t+1};\theta_{t+1})^{-\alpha} V_{11}(a^*_{t+1};\theta_{t+1}) - \alpha\, V(a^*_{t+1};\theta_{t+1})^{-\alpha-1} V_1(a^*_{t+1};\theta_{t+1})^2\right]}{E_t\ V(a^*_{t+1};\theta_{t+1})^{-\alpha} V_1(a^*_{t+1};\theta_{t+1})}$$

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Evaluated at the nonstochastic steady state, this simplifies to:

$$R^a(a; \theta) \; = \; \frac{-V_{11}(a; \theta)}{V_1(a; \theta)} + \alpha \, \frac{V_1(a; \theta)}{V(a; \theta)} \, .$$

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Folk wisdom ($\alpha = 0$): Constantinides (1990), Farmer (1990), Campbell-Cochrane (1999), Boldrin-Christiano-Fisher (1997, 2001), Flavin-Nakagawa (2008)

Solve for V_1 and V_{11}

Benveniste-Scheinkman:

$$V_1(a_t; \theta_t) = (1 + r_t)\, u_1(c_t^*, l_t^*). \qquad (*)$$

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Differentiate $(*)$ to get:

$$V_{11}(a_t; \theta_t) \;=\; (1 + r_t) \left[u_{11}(c_t^*, l_t^*) \frac{\partial c_t^*}{\partial a_t} \;+\; u_{12}(c_t^*, l_t^*) \frac{\partial l_t^*}{\partial a_t} \right].$$

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Intratemporal optimality: $\dfrac{\partial l_t^*}{\partial a_t} = -\lambda \dfrac{\partial c_t^*}{\partial a_t}, \qquad \lambda = \frac{wu_{11}+u_{12}}{u_{22}+wu_{12}}$

Euler equation and BC: $\dfrac{\partial c_t^*}{\partial a_t} = \dfrac{r}{1+w\lambda}.$

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Proposition 3. *The household's coefficient of absolute risk aversion in Proposition 1, evaluated at steady state, satisfies:*

$$R^a(a;\theta) \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda} \;+\; \alpha\frac{r\,u_1}{u}.$$

Relative vs. Absolute Risk Aversion

Relative risk aversion depends on household wealth.

Household wealth includes:

- financial assets a_t
- present value of nonlabor income, d_t
- present value of labor income, $w_t l_t$
- maybe present value of leisure, $w_t(\bar{l} - l_t)$?

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- maybe present value of leisure, $w_t(\bar{l} - l_t)$?

Leisure can be hard to define, e.g.,

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}$$

Two Coefficients of Relative Risk Aversion

Definition 2. *The consumption-wealth coefficient of relative risk aversion,* $R^c(a_t;\theta_t) \equiv A_t^c R^a(a_t;\theta_t)$*, where* A_t^c *denotes the present discounted value of household consumption.*

At steady state:

$$R^c(a;\theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda} + \alpha \frac{cu_1}{u}.$$

Definition 3. *The consumption-and-leisure-wealth coefficient of relative risk aversion,* $R^{cl}(a_t;\theta_t) \equiv A_t^{cl} R^a(a_t;\theta_t)$*, where* A_t^{cl} *denotes the present discounted value of consumption and leisure.*

At steady state:

$$R^{cl}(a;\theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1 + w\lambda} + \alpha \frac{\big(c + w(\bar{l} - l)\big)u_1}{u}.$$

Asset Pricing

Expected excess return on asset i:

$$\begin{aligned} \psi_t^i &\equiv E_t r_{t+1}^i - r_{t+1}^f \\ &= -\mathrm{Cov}_t(m_{t+1}, r_{t+1}^i) \end{aligned}$$

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Proposition 7. *To first order around the nonstochastic steady state,*

$$dm_{t+1} = -R^a(a;\theta)\, d\hat{A}_{t+1} + d\Phi_{t+1}$$

To second order around the nonstochastic steady state,

$$\psi_t^i = R^a(a;\theta)\, Cov_t(dr_{t+1}^i, d\hat{A}_{t+1}) - Cov_t(dr_{t+1}^i, d\Phi_{t+1})$$

Numerical Example

Economy is a very simple, standard RBC model:

- Competitive firms
- Cobb-Douglas production, $y_t = Z_t k_t^{1-\zeta} l_t^{\zeta}$
- AR(1) technology, $\log Z_{t+1} = \rho_z \log Z_t + \varepsilon_t$
- Capital accumulation, $k_{t+1} = (1-\delta)k_t + y_t - c_t$
- Equity is a consumption claim
- Equity premium is expected excess return,

$$\psi_t = \frac{E_t(C_{t+1} + p_{t+1})}{p_t} - (1 + r_t^f)$$

Numerical Example: Preferences

Period utility

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}$$

Generalized recursive preferences

$$V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta \left(E_t \, V(a_{t+1}; \theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)}$$

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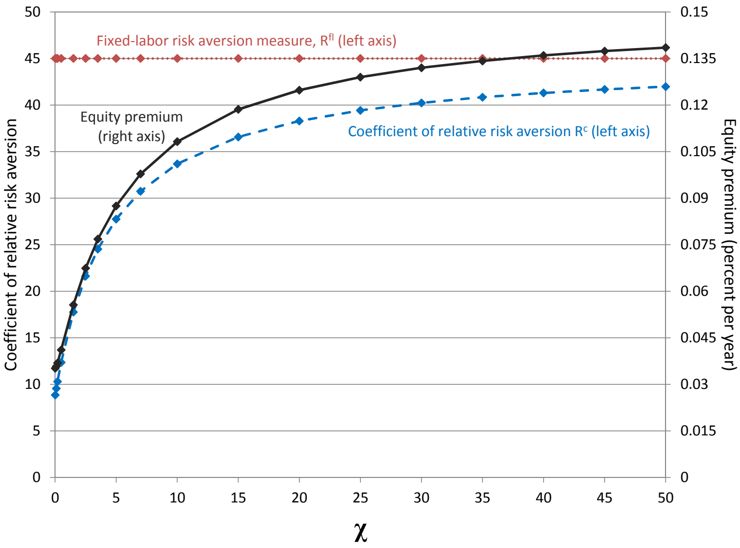
Note:

- IES $= 1/\gamma$
- If labor fixed, relative risk aversion is $R^{fl} = \gamma + \alpha(1-\gamma)$
- Epstein-Zin, Weil define $\widetilde{\alpha} = \gamma + \alpha(1-\gamma)$
- If labor flexible, relative risk aversion is R^c, depends on χ, γ, α

Additively Separable Period Utility



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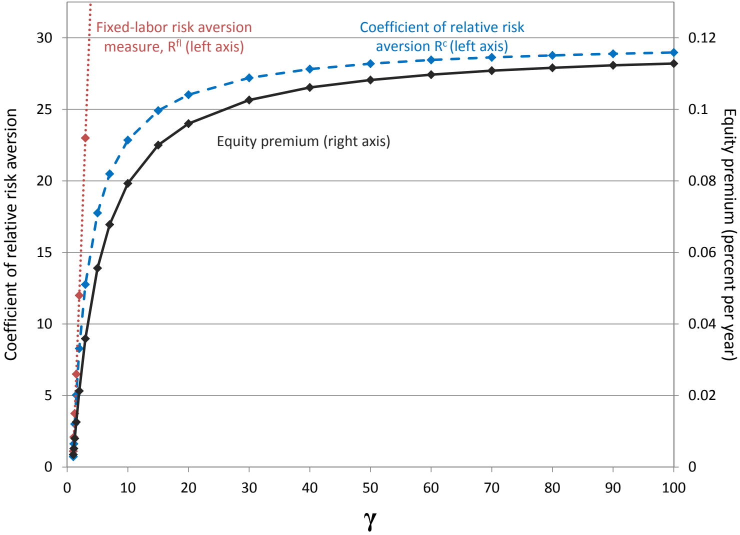


Second Numerical Example

Same RBC model as before, with Cobb-Douglas period utility

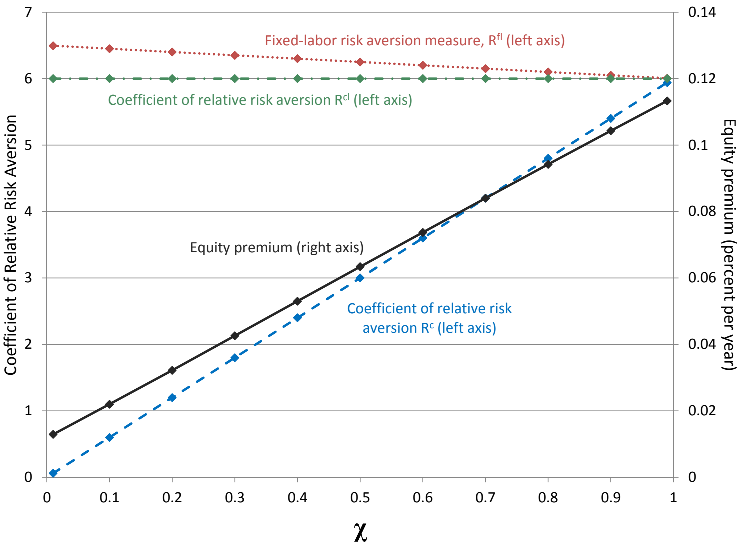
$$u(c_t, l_t) = \frac{\left(c_t^{\chi}(1-l_t)^{1-\chi}\right)^{1-\gamma}}{1-\gamma}$$

and random-walk technology, $\rho_z = 1$.

Note:

- IES $= 1/\gamma$
- If labor fixed, risk aversion is $R^{fl} = (1-\chi(1-\gamma)) + \alpha(1-\gamma)$
- For composite good, risk aversion is $R^{cl} = \gamma + \alpha(1-\gamma)$
- Epstein-Zin-Weil consider $\chi = 1$, define $\widetilde{\alpha} = \gamma + \alpha(1-\gamma)$
- Risk aversion R^c recognizes labor is flexible, excludes value of leisure from household wealth, $R^c = \chi\gamma + \chi\alpha(1-\gamma)$

Cobb-Douglas Period Utility



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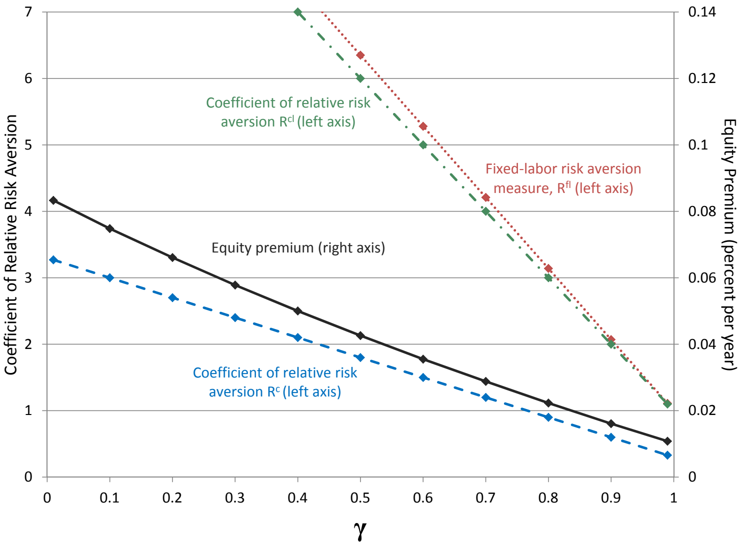

Conclusions

1. A flexible labor margin affects risk aversion
2. Risk premia are related to risk aversion
3. Fixed-labor and composite-good measures of risk aversion perform poorly
4. For multiplier preferences, risk aversion is very sensitive to scaling by $(1-\beta)$
5. Simple, closed-form expressions for risk aversion with:
   - flexible labor margin
   - generalized recursive preferences
   - external or internal habits
   - validity away from steady state
   - correspondence to risk premia in the model
6. Ongoing work: frictional labor markets